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Current Exchanges and Unconstrained Higher Spins

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Abstract

The (Fang-)Fronsdal formulation for free fully symmetric (spinor-) tensors rests on (γ-)trace constraints on gauge fields and parameters. When these are relaxed, glimpses of the underlying geometry emerge: the field equations extend to non-local expressions involving the higher-spin curvatures, and with only a pair of additional fields an equivalent "minimal" local formulation is also possible. In this paper we complete the discussion of the "minimal" formulation for fully symmetric (spinor-) tensors, constructing one-parameter families of Lagrangians and extending them to $(A)dS$ backgrounds. We then turn on external currents, that in this setting are subject to conventional conservation laws and, by a close scrutiny of current exchanges in the various formulations, we clarify the precise link between the local and non-local versions of the theory. To this end, we first show the equivalence of the constrained and unconstrained local formulations, and then identify a unique set of non-local Lagrangian equations which behave in the same fashion in current exchanges.

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Contents

1 Introduction 3

2 Minimal Lagrangians for unconstrained higher spins 6

2.1 Bosons 6

2.2 Fermions 10

3 Current exchanges in the local formulations 14

3.1 Bose fields in flat space 15

3.2 Fermi fields in flat space 20

3.3 Current exchanges in an AdS background 22

4 Current exchanges in the non-local formulation 23

4.1 The problem 23

4.2 Nonlocal equations with a proper current exchange 27

5 Conclusions 32

A Notation and conventions 34

1 Introduction

The (Fang-)Fronsdal formulation of free higher-spin dynamics [1] rests on trace (or γ-trace) constraints for gauge fields and corresponding gauge parameters[2]. While these algebraic conditions do not conflict with Lorentz covariance, it is quite natural to try and forego them, aiming at formulations that are closer in spirit to the familiar ones for low-spin fields. The issues are then how to do it and what one gains from this extension.

A direct, if somewhat unconventional, way to forego the trace constraints of [1] is via a set of non-local equations involving the higher-spin curvatures of de Wit and Freedman [6]. This was first done in [7] for the set of higher-spin fields originally considered in [1], totally symmetric tensors $\varphi_{\mu_1 \ldots \mu_s}$, that here will be loosely referred to as spin-s fields, and for corresponding totally symmetric tensor-spinors $\psi_{\mu_1 \ldots \mu_n}$, that here will be loosely referred to as spin-$(n+1/2)$ fields. In exploring new directions for higher spins, it is customary to begin by restricting the attention to this class of fields, although they do not exhaust the available choices in $D > 4$, since the resulting analysis suffices to display some key features of the problem in a relatively handy setting. The generalization to tensors of mixed symmetry is then an important further step, and is also crucial for establishing a proper link with String Theory. For the non-local formulation this was first achieved to some extent in [8], and the resulting properties were recently explored in further detail in [9].

A less direct, but more conventional, way to forego the trace constraints is via the introduction of compensators. This was first achieved by Pashnev, Tsulaia, Buchbinder and others [10], using a BRST [11] formulation inspired by String Field Theory but adapted to the description of individual massless higher-spin modes. This is to be contrasted with the conventional BRST formulation of String Field Theory [12], that associates to a given higher-spin field a whole family of other fields belonging to lower Regge trajectories. The BRST formulation of individual higher-spin fields of [10] is rather complicated, and makes use of $\mathcal{O}(s)$ fields to describe the propagation of a single set of spin-s modes. On the other hand, it is possible to present "compensator equations" for individual higher-spin fields that, while non-Lagrangian, are fully gauge invariant without any trace constraints and involve only two fields. For instance, in the bosonic case, aside from $\varphi_{\mu_1 \ldots \mu_s}$ they involve a single compensator field $\alpha_{\mu_1 \ldots \mu_{s-3}}$, which first emerges for $s = 3$. The relation with free String Field Theory was clarified in [13], where the precise link between the compensator equations for higher-spin fields and the "triplet" systems [14, 13, 15] emerging from free String Field Theory in the low-tension limit was displayed. In [16] these equations were then related to the results in [10], from which they can be recovered by a partial gauge fixing, and were also extended to $(A)dS$ backgrounds. Interestingly, the simplest flat-space equation in this set, for a spin-3 field $\varphi_{\mu\nu\rho}$ and a scalar compensator α,

$$\Box \varphi_{\mu\nu\rho} - (\partial_\mu \partial \cdot \varphi_{\nu\rho} + \ldots) + (\partial_\mu \partial_\nu \varphi'_\rho + \ldots) = 3\, \partial_\mu \partial_\nu \partial_\rho \alpha \ , \tag{1.1}$$

where, as in the rest of the present paper, "primes" denote traces, was first considered by Schwinger long ago [17] [3]. More recently, a "minimal" Lagrangian formulation for the compensator equations was also obtained in [18]: rather than $\mathcal{O}(s)$ fields as in [10], it involves only a Lagrange multiplier $\beta_{\mu_1 \ldots \mu_{s-4}}$, that first emerges for spin $s = 4$, aside from the basic field $\varphi_{\mu_1 \ldots \mu_s}$ and the compensator $\alpha_{\mu_1 \ldots \mu_{s-3}}$. In contrast with the non-local case, the generalization of this

[2] The web site http://www.ulb.ac.be/sciences/ptm/pmif/Solvay1proc.pdf contains the Proceedings of the First Solvay Workshop on Higher-Spin Gauge Theories [2], with some contributions closely related to the present work [3, 4, 5] and many references to the original literature.

[3] We are grateful to G. Savvidy for calling Schwinger's result to our attention.

"minimal" local formulation to tensors of mixed symmetry is not known at the present time, although its key features may be anticipated from the known constrained gauge transformations.

While it is certainly interesting and instructive to explore them, it is difficult to assess the relative virtues of different formulations for free higher-spin fields before the systematics of higher-spin interactions is further clarified. There have been many attempts in this direction over the years [19], that have marked the history of the subject as a result of the unexpected difficulties that were readily met, but this line of approach deserves further efforts and is being further explored, with the help of more powerful techniques, in recent times [20]. It is fair to say that we do not possess yet a general understanding of higher-spin interactions and of the underlying geometry, but we do have at our disposal two important paradigmatic examples that are based on a well motivated algebraic setting. These are the Vasiliev equations, in their four-dimensional form based on spinor oscillators [21] and in their more recent D-dimensional form based on vector oscillators [22]. The Vasiliev equations are a set of first-order differential constraints involving a one-form master field A, that encodes an infinite family of $\varphi_{\mu_1 \ldots \mu_s}$ via corresponding generalized vielbeins and spin connections, and a zero-form master field Φ, that collects their Weyl curvatures and covariant derivatives, together with corresponding data for a scalar mode. Due to the presence of the zero-form Φ, the Vasiliev constraints generalize the more conventional notion of free-differential algebra [23] in a non-trivial fashion, and embody a description of free higher-spin modes in an "unfolded" form, via infinitely many auxiliary fields that subsume their local data. Both the four-dimensional formulation of [21] and the D-dimensional formulation of [22] are not Lagrangian, but while the former is fully based on the constrained Fronsdal form of the free dynamics, this is not quite true for the latter. More precisely, the spinor oscillators build consistent interactions for the "frame" version of Fronsdal's formulation, which was first discussed in [24], and can be gauge-fixed to its "metric" version but does not leave room for the compensator of [13, 16, 18] or for the wider gauge symmetry of [10]. On the other hand, the field equations of [22], as pointed out in [5], allow a non-dynamical "off-shell" variant devoid of trace constraints and, if completed with a "strong" projection, turn into dynamical equations that at the free level reduce precisely to a frame version of the compensator equations of [13, 16, 18]. The potential open problems with the resulting interactions are still not fully sorted out at the present time, and we refer the reader to [5, 4] for further details, but there are reasons to believe that fully interacting equations can be defined in this fashion, thus encoding naturally a gauge symmetry not subject to Fronsdal's trace constraints.

To summarize, even with our current incomplete grasp of the systematics of higher-spin interactions the unconstrained formulation of free higher-spin fields presents a number of attractive features. First, and most notably, it embodies a direct link with higher-spin geometry, since its minimal, non-local form of [7] rests directly on the higher-spin curvatures of [6] rather than on lower connections. In addition, it links naturally to the BRST form of free String Field Theory, from which it emerges, after a suitable truncation to the leading Regge trajectory, in the low-tension limit [13, 16]. Finally, and most importantly for the purposes of the present paper, it couples to *conserved* currents, to be contrasted with the partially conserved currents of the Fang-Fronsdal formulation.

This paper is devoted to exploring some key features of the "minimal" unconstrained Lagrangian formalism of [18] that manifest themselves in the presence of external currents. However, we begin in Section 2 by reconsidering and simplifying somewhat the results of [18] for bosons and fermions, that here are also extended to the interesting cases of $(A)dS$ backgrounds.

While the extension of flat-space results to these more general backgrounds is an interesting and well-defined problem in its own right, this is perhaps the place to spend, as in [5], some words of caution against the naive identification of the spin-2 modes of the Vasiliev equations with the gravitational field. The problem arises since, by virtue of their very field content, the Vasiliev equations are naturally an effective description of the first Regge trajectory of the *open* bosonic string, albeit in an unconventional and largely unexplored regime where it would experience a gigantic collapse of all its massive modes to the massless level. It should be stressed that little is presently known about this stringy regime, although the AdS/CFT correspondence, in the weak gauge-coupling limit, provides indirect arguments for it [25]. The Vasiliev equations also allow a Chan-Paton [26] extension to matrix-valued modes, precisely as is the case for the open bosonic string. Hence, they generally involve not a single "graviton", but rather a whole multiplet of spin-2 modes, so that the "minimal" Vasiliev model, whose interest was clearly stressed in [27] and whose first non-trivial "cosmological" solution was recently presented in [28], is somehow the counterpart in this context of the $O(1)$ open string discussed by Schwarz in his 1982 review [29], whose open sector contains indeed only even excitation levels. Should one then associate the singlet spin-2 Vasiliev mode with gravity, that lies outside the open spectrum? Our present knowledge does not allow sharp statements to this effect, and caution is called for. Indeed, while the distinction between open and closed spectra is strictly in place for tensile strings, in the largely unexplored low-tension limit a mixing between string states at different levels, and in particular between the singlet part of the spin-2 open-string field and the closed-string graviton, might take place, although a clear understanding of this important issue from a string vantage point is lacking at present.

In a similar spirit, a closed-string analogue of the Vasiliev equations, which should rest on a more subtle algebraic structure in order not to allow a Chan-Paton extension, is not available at the present time, and the search for it might provide important clues on low-tension string regimes. These are clearly deep issues on which, unfortunately, we shall have little to say in the present paper. Taking a close look at current exchanges, however, in Sections 3 and 4 we shall be able to investigate in some detail to which extent the available options for formulating the free higher-spin dynamics lead to the same counting of degrees of freedom. Thus, in Section 3 we shall establish the direct equivalence between the constrained Fronsdal formulation of [1] and the unconstrained local formulation of [18]. On the other hand, the nonlocal counterpart of the unconstrained free theory is not fully determined. Many possible options for a non-local spin-s Lagrangian equation exist and, as we shall see, the simple choice made in [7] for an Einstein-like tensor *does not* provide the direct counterpart of the local Lagrangian equations of [18]. In Section 4, however, we shall identify a different, unique form of the non-local Lagrangian field equations which behaves exactly like the local formulations, thus arriving at a precise link between the local and non-local unconstrained forms of free higher-spin gauge theory. Let us stress that the issue here is the correct coupling to external currents via the natural source term, $\varphi \cdot J$, and the Lagrangian equations that couple correctly to external currents will be more complicated than those proposed in [7]. Still, in the absence of external currents they can be turned into the non-local non-Lagrangian equations of [7],

$$\frac{1}{\Box^{\,p}}\, \partial \cdot \mathcal{R}^{\,[p]}{}_{;\,\alpha_1 \cdots \alpha_{2p+1}} \;=\; 0 \tag{1.2}$$

for odd spins $s = 2p + 1$, and

$$\frac{1}{\Box^{\,p-1}}\, \mathcal{R}^{\,[p]}{}_{;\,\alpha_1 \cdots \alpha_{2p}} \;=\; 0 \tag{1.3}$$

for even spins $s = 2p$. Section 5 contains our Conclusions, while the Appendix collects some useful results concerning our implicit notation for symmetric tensors and our conventions.

2 Minimal Lagrangians for unconstrained higher spins

In this Section we review the construction of the minimal unconstrained free Lagrangians for fully symmetric higher-spin tensors and tensor-spinors in flat space presented in [18]. Our aim is to streamline both the derivation and the resulting presentation, by properly stressing the role of a few gauge invariant constructs playing the role of "building blocks" for the dynamical quantities of interest. As we shall see, the resulting simplified form of the Lagrangians proves quite helpful in extending the previous results to the interesting cases of $(A)dS$ backgrounds.

2.1 Bosons

The definition of fully gauge invariant kinetic operators is a very convenient starting point for the construction of the minimal bosonic Lagrangians, that we would like to reconsider and extend here. For a rank-s fully symmetric tensor, that in the index-free notation of [18] (see the Appendix for details and some examples) can be simply denoted by φ, one can begin by considering the Fronsdal operator

$$\mathcal{F} \,=\, \Box\,\varphi \,-\, \partial\,\partial\cdot\varphi \,+\, \partial^2\varphi\,' \,, \tag{2.1}$$

where, here as elsewhere in this paper, a "prime" denotes a trace. Under the gauge transformation

$$\delta\,\varphi \,=\, \partial\,\Lambda\,, \tag{2.2}$$

$\mathcal{F}$ varies according to

$$\delta\,\mathcal{F} \,=\, 3\,\partial^{\,3}\,\Lambda'\,, \tag{2.3}$$

where, as anticipated, Λ' denotes the trace of the gauge parameter Λ. From $\mathcal{F}$ one can build the *fully* gauge invariant tensor

$$\mathcal{A} \,=\, \mathcal{F} \,-\, 3\,\partial^{\,3}\,\alpha\,, \tag{2.4}$$

where the spin-$(s-3)$ *compensator* α transforms as

$$\delta\,\alpha \,=\, \Lambda' \tag{2.5}$$

under the tensor gauge transformation in eq. (2.2), and one can then show that $\mathcal{A}$ satisfies the Bianchi identity

$$\partial\cdot\mathcal{A} \,-\, \frac{1}{2}\,\partial\,\mathcal{A}' \,=\, -\,\frac{3}{2}\,\partial^{\,3}\,\left(\varphi'' \,-\, 4\,\partial\cdot\alpha \,-\, \partial\,\alpha'\right)\,. \tag{2.6}$$

With the "mostly-positive" space-time signature that we shall use throughout, the minimal bosonic Lagrangians of [18] can thus be conveniently recovered starting from

$$\mathcal{L}_0 \,=\, \frac{1}{2}\,\varphi\,\left(\mathcal{A} \,-\, \frac{1}{2}\,\eta\,\mathcal{A}\,'\right)\,, \tag{2.7}$$

which, on account of (2.1) and (2.6), varies as

$$\delta\mathcal{L}_0 \,=\, \frac{3}{4}\,\binom{s}{3}\,\Lambda'\partial\cdot\mathcal{A}' \,-\, 3\,\binom{s}{4}\,\partial\cdot\partial\cdot\partial\cdot\Lambda\,\left[\varphi'' \,-\, 4\,\partial\cdot\alpha \,-\, \partial\,\alpha'\right] \tag{2.8}$$

under the tensor gauge transformations (2.2) and (2.5). These terms can be compensated adding

$$\mathcal{L}_1 \,=\, -\,\frac{3}{4}\,\binom{s}{3}\,\alpha\,\partial\cdot\mathcal{A}' \,+\, 3\,\binom{s}{4}\,\beta\,\left[\varphi'' \,-\, 4\,\partial\cdot\alpha \,-\, \partial\,\alpha'\right]\,, \tag{2.9}$$

so that the end result can be presented in the rather compact form

$$\mathcal{L} \,=\, \frac{1}{2}\,\varphi\,\left(\mathcal{A}\,-\,\frac{1}{2}\,\eta\,\mathcal{A}\,'\right)\,-\,\frac{3}{4}\,\binom{s}{3}\,\alpha\,\partial\cdot\mathcal{A}\,'\,+\,3\,\binom{s}{4}\,\beta\,\left[\varphi\,''\,-\,4\,\partial\cdot\alpha\,-\,\partial\,\alpha'\right]\,, \tag{2.10}$$

where, as in [18], the *Lagrange multiplier* β transforms as

$$\delta\beta \,=\, \partial\cdot\partial\cdot\partial\cdot\Lambda\,. \tag{2.11}$$

We would like to stress that working insofar as possible with the gauge-invariant tensor $\mathcal{A}$ has streamlined both the derivation and the final form of the Lagrangian (2.10) with respect to [18]. One can actually do better defining, for the bosonic case, three independent gauge-invariant tensors in terms of which all dynamical quantities of interest can be expressed. The first member of this set is the tensor $\mathcal{A}$ introduced in (2.4), the second,

$$\mathcal{C} \,\equiv\, \varphi''\,-\,4\,\partial\cdot\alpha\,-\,\partial\,\alpha\,'\,, \tag{2.12}$$

defines the constraint, and finally the third,

$$\mathcal{B} \,\equiv\, \beta\,-\,\frac{1}{2}\,(\partial\cdot\partial\cdot\varphi'\,-\,2\,\Box\,\partial\cdot\alpha\,-\,\partial\,\partial\cdot\partial\cdot\alpha)\,, \tag{2.13}$$

relates the Lagrange multiplier β to the single combination of φ and α that possesses an identical gauge transformation. Actually, the very definition of $\mathcal{B}$ suggests that the Lagrangian (2.10) is not the only possibility, and may be generalized by turning the coefficient of $\mathcal{C}$ into

$$\mathcal{L}_k \,=\, \frac{1}{2}\,\varphi\,\left(\mathcal{A}\,-\,\frac{1}{2}\,\eta\,\mathcal{A}\,'\right)\,-\,\frac{3}{4}\,\binom{s}{3}\,\alpha\,\partial\cdot\mathcal{A}\,'\,+\,3\,\binom{s}{4}\,[\beta\,-\,k\,\mathcal{B}]\,\mathcal{C}\,, \tag{2.14}$$

so that (2.10) corresponds to the particular choice $k\,=\,0$.

A more general analysis should involve additional quadratic terms mixing $\mathcal{A}$ and $\mathcal{C}$ tensors, as well as terms quadratic in the tensor $\mathcal{C}$, thus exhausting all possible gauge-invariant terms with at most two derivatives for the physical field φ and at most four derivatives for the compensator α. In particular, given the identity

$$\mathcal{A}''\,=\,3\,\Box\,\mathcal{C}\,+\,3\,\partial\,\partial\cdot\mathcal{C}\,+\,\partial^{\,2}\,\mathcal{C}\,'\,, \tag{2.15}$$

and since the only possibility to combine $\mathcal{A}$ and $\mathcal{C}$ without increasing the number of derivatives in φ is via $\mathcal{A}''\,\mathcal{C}$, it turns out that the only possibilities left are quadratic terms in the tensor $\mathcal{C}$,

$$\mathcal{C}^{\,2}\,,\qquad \mathcal{C}\,\Box\,\mathcal{C}\,,\qquad \partial\cdot\mathcal{C}\,\partial\cdot\mathcal{C}\,,\qquad \mathcal{C}\,'\,\partial\cdot\partial\cdot\mathcal{C}\,, \tag{2.16}$$

together with their traces. Nonetheless, all these terms would only generate additional terms linear in $\mathcal{C}$ in the equations for φ and α whose role would be irrelevant once $\mathcal{C}$ is set to zero by the field equation for β.

We have thus arrived at a one-parameter family of gauge-invariant, unconstrained Lagrangians, whose field equations can be nicely expressed in terms of the three tensors $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, and read

$$\begin{aligned} &E_\varphi(k) \equiv \mathcal{A}\,-\,\frac{1}{2}\,\eta\,\mathcal{A}\,'\,+\,\frac{1}{4}\,(1\,+\,k)\,\eta\,\partial^{\,2}\,\mathcal{C}\,+\,(1\,-\,k)\,\eta^{\,2}\,\mathcal{B}\,=\,0\,,\\ &E_\alpha(k) \equiv\,-\,\frac{3}{2}\binom{s}{3}\{\partial\cdot\mathcal{A}\,'\,-\,\frac{k\,+\,1}{2}\,(\partial\,\Box\,+\,\partial^{\,2}\,\partial\cdot)\,\mathcal{C}\,+(k\,-\,1)\,(2\,\partial\,+\,\eta\,\partial\cdot)\,\mathcal{B}\}\,=\,0\,,\\ &E_\beta(k) \equiv\,3\,\binom{s}{4}\,(1\,-\,k)\,\mathcal{C}\,=\,0\,, \end{aligned} \tag{2.17}$$

where it should be noted that, in terms of the tensors (2.17), eq. (2.14) can be cast in the particularly compact form

$$\mathcal{L}_k \,=\, \frac{1}{2}\,\varphi\, E_\varphi(k) \,+\, \frac{1}{2}\,\alpha\, E_\alpha(k) \,+\, \frac{1}{2}\,\beta\, E_\beta(k)\,. \tag{2.18}$$

For generic values of k the field equation for φ can be reduced to the compensator form $\mathcal{A} \,=\, 0$, while $\mathcal{B} = 0$ determines the Lagrange multiplier β in terms of the other fields, as was shown to be case in [18] for $k = 0$. One is eventually left with the non-Lagrangian compensator equations of [13, 16],

$$\begin{aligned} &\mathcal{A} \equiv \mathcal{F} \,-\, 3\,\partial^{\,3}\,\alpha \,=\, 0\,, \\ &\mathcal{C} \equiv \varphi'' \,-\, 4\,\partial\cdot\alpha \,-\, \partial\,\alpha' \,=\, 0\,, \end{aligned} \tag{2.19}$$

and these can be finally gauge-fixed to the Fronsdal form $\mathcal{F} \,=\, 0$ and to Fronsdal's double-trace constraint $\varphi'' \,=\, 0$, making use of the trace Λ' of the gauge parameter to remove the compensator α. This indicates that the unconstrained Lagrangians (2.14) provide a proper description of higher-spin dynamics, just like Fronsdal's constrained formulation.

The value $k = -1$ appears particularly interesting, since in this case the tensor $\mathcal{C}$ disappears from the field equations for φ and α. Finally, the choice $k = +1$ is somewhat degenerate, since in this case β disappears from the Lagrangian, and this implies, in its turn, that $\mathcal{B}$ disappears from the field equations for φ and α. Whereas naively this could be seen as a simplification, in this case one can only derive directly from the field equations the condition

$$\Box\,\mathcal{C} \,=\, 0\,, \tag{2.20}$$

which is not sufficient to remove the double trace.

In all cases, however, the divergence of $E_\varphi(k)$ vanishes when α and β are on-shell, as demanded by the Noether relation

$$\partial\cdot E_\varphi(k) \,=\, \frac{1}{3\,\binom{s}{3}}\,\eta\, E_\alpha(k) \,-\, \frac{1}{4\,\binom{s}{4}}\,\partial^{\,3}\, E_\beta(k)\,, \tag{2.21}$$

that implies that external sources coupled to the dynamical field φ are to be *conserved*.

As anticipated, this presentation of the Lagrangians has the virtue of leading rather directly to their $(A)dS$ deformations. These draw their origin from the basic $(A)dS$ commutator for two covariant derivatives acting on a vector field V, that reads

$$[\,\nabla_\mu\,,\nabla_\nu\,]\;V_\rho \,=\, \frac{1}{L^2}\,(g_{\nu\rho}V_\mu \,-\, g_{\mu\rho}V_\nu)\;, \tag{2.22}$$

where L denotes the $(A)dS$ radius. For brevity, in the following we shall refer only to the AdS case, since the corresponding results for dS backgrounds can be simply obtained by the formal continuation of L to imaginary values.

In discussing higher-spin gauge fields in these curved backgrounds, it is convenient to begin by considering the deformed AdS Fronsdal operator [30] (see also, for instance, [16])

$$\mathcal{F}_L \,=\, \mathcal{F} \,-\, \frac{1}{L^2}\,\left\{[(3-D-s)(2-s)-s]\;\varphi \,+\, 2\,g\,\varphi^{'}\right\}\,, \tag{2.23}$$

where D denotes the space-time dimension and

$$\mathcal{F} \,=\, \Box\,\varphi \,-\, \nabla\,\nabla\cdot\varphi \,+\, \nabla^{\,2}\,\varphi' \tag{2.24}$$

is the AdS-covariantized Fronsdal operator, that transforms according to

$$\delta\mathcal{F}_L \,=\, 3\,\nabla^{\,3}\Lambda' \,-\, \frac{4}{L^2}\,g\,\nabla\,\Lambda' \tag{2.25}$$

under the gauge transformation

$$\delta\varphi \,=\, \nabla\Lambda\ . \tag{2.26}$$

Even in this more general AdS setting the structure of the theory can be fully encoded in three gauge invariant tensors, $\mathcal{A}_L$, $\mathcal{B}_L$ and $\mathcal{C}_L$, that reduce to the previous expressions in the flat limit. Assuming for the auxiliary fields the straightforward generalisations of the gauge transformations (2.5), (2.11),

$$\begin{aligned} \delta\,\alpha \,&=\, \Lambda' \\ \delta\,\beta \,&=\, \nabla\cdot\nabla\cdot\nabla\cdot\Lambda\,, \end{aligned} \tag{2.27}$$

the AdS generalizations of $\mathcal{A}$ and $\mathcal{C}$,

$$\mathcal{A}_L \,=\, \mathcal{F}_L \,-\, 3\,\nabla^{\,3}\alpha \,+\, \frac{4}{L^2}\,g\,\nabla\,\alpha\,, \tag{2.28}$$

$$\mathcal{C}_L \,=\, \varphi'' \,-\, 4\,\nabla\cdot\alpha \,-\, \nabla\,\alpha'\,, \tag{2.29}$$

are rather simple, while the covariantization of $\mathcal{B}$,

$$\begin{aligned} \mathcal{B}_L \,=\, \beta \,-\, \Big\{ & \frac{1}{2}\nabla\cdot\nabla\cdot\varphi' \,-\, \Box\,\nabla\cdot\alpha \,-\, \frac{1}{2}\,\nabla\nabla\cdot\nabla\cdot\alpha \\ & -\, \frac{1}{L^{\,2}}\left(2\,\nabla\alpha' \,+\, 2\,g\,\nabla\cdot\alpha' \,+\, [(s\,-\,3)(5\,-\,s\,-\,D)]\,\nabla\cdot\alpha\right)\Big\} \end{aligned} \tag{2.30}$$

is somewhat more tedious to obtain, since new terms appear in the gauge variation of $\nabla\cdot\nabla\cdot\varphi'$ in this curved background.

The starting point for the construction of the bosonic Lagrangians is now

$$\mathcal{L}_0 \,=\, \frac{e}{2}\,\varphi\,\left(\mathcal{A}_L \,-\, \frac{1}{2}\,g\,\mathcal{A}_L'\right)\,, \tag{2.31}$$

where e and g denote the determinant of the vielbein and the AdS metric. Proceeding as in flat space and taking into account the deformed Bianchi identity,

$$\nabla\cdot\mathcal{A}_L \,-\, \frac{1}{2}\,\nabla\,\mathcal{A}_L' = -\,\frac{3}{2}\,\nabla^{\,3}\,\mathcal{C}_L \,+\, \frac{2}{L^{\,2}}\,g\,\nabla\,\mathcal{C}_L\,, \tag{2.32}$$

one can finally arrive at the gauge invariant Lagrangians

$$\mathcal{L} \,=\, \frac{e}{2}\,\varphi\,\left(\mathcal{A}_L \,-\, \frac{1}{2}\,g\,\mathcal{A}_L'\right) \,-\, \frac{3e}{4}\,\binom{s}{3}\,\alpha\,\nabla\cdot\mathcal{A}_L' \,+\, 3\,e\,\binom{s}{4}\,[\beta \,-\, \frac{4}{L^{\,2}}\,\nabla\cdot\alpha]\,\mathcal{C}_L\,. \tag{2.33}$$

that, as in the flat case, are special members of a whole family of gauge invariant Lagrangians,

$$\mathcal{L}_k \,=\, \frac{e}{2}\,\varphi\,\left(\mathcal{A}_L \,-\, \frac{1}{2}\,g\,\mathcal{A}_L'\right) \,-\, \frac{3e}{4}\,\binom{s}{3}\,\alpha\,\nabla\cdot\mathcal{A}_L' + 3\,e\,\binom{s}{4}\,\left[\beta \,-\, \frac{4}{L^{\,2}}\,\nabla\cdot\alpha \,-\, k\,\mathcal{B}_L\right]\,\mathcal{C}_L\,, \tag{2.34}$$

defined introducing in $\mathcal{L}$ the gauge-invariant coupling $\mathcal{B}_L\,\mathcal{C}_L$. From these Lagrangians one can then derive the AdS field equations: for the fields φ and β, they have the form of those obtained in flat case, aside from the natural substitutions

$$(\mathcal{A}\,,\mathcal{B}\,,\mathcal{C}\,,\eta\,,\partial)\ \longrightarrow\ (\mathcal{A}_L\,,\mathcal{B}_L\,,\mathcal{C}_L\,,g\,,\nabla)\,, \tag{2.35}$$

while the field equation for α contains additional terms that depend on the tensor $\mathcal{C}_L$. The final result reads

$$\begin{aligned}
&E_{\varphi\,L}(k) \equiv\ \mathcal{A}_L\ -\ \frac{1}{2}\,g\,\mathcal{A}_L'\ +\ \frac{1}{4}\,(1\,+\,k)\,g\,\nabla^{\,2}\,\mathcal{C}_L\ +\ (1\,-\,k)\,g^{\,2}\,\mathcal{B}_L\ =\ 0\,,\\
&E_{\alpha\,L}(k) \equiv\ -\frac{3}{2}\binom{s}{3}\left\{\nabla\cdot\mathcal{A}_L'\ -\ \frac{k\,+\,1}{2}\,\nabla\cdot(\nabla^{\,2}\,\mathcal{C}_L)\ +(k\,-\,1)\,(2\,\nabla\ +\ g\,\nabla\cdot)\,\mathcal{B}_L\ -\ \frac{4}{L^{\,2}}\nabla\,\mathcal{C}_L\right\}\ =\ 0\,,\\
&E_{\beta\,L}(k) \equiv\ 3\binom{s}{4}\,(1\,-\,k)\,\mathcal{C}_L\ =\ 0\,.
\end{aligned} \tag{2.36}$$

More explicitly, in the case $k\,=\,0$ the field equation for φ takes the form

$$\begin{aligned}
E_{\varphi\,L}\ \equiv&\,\mathcal{A}_L\ -\ \frac{1}{2}\,g\left\{\mathcal{A}_L'\ -\ \frac{1}{2}\nabla^{\,2}\left(\varphi''-4\,\nabla\cdot\alpha\ -\ \nabla\,\alpha'\right)\right\}\\
&+\ g^2\left\{\beta\ +\ \frac{1}{2}\nabla\nabla\cdot\nabla\cdot\alpha\ +\ \Box\nabla\cdot\alpha\ -\ \frac{1}{2}\ \nabla\cdot\nabla\cdot\varphi'\ +\ \frac{2}{L^2}\ \nabla\alpha'\right.\\
&\left.+\ \frac{2}{L^2}\ g\,\nabla\cdot\alpha'\ +\ \frac{1}{L^2}\ [(s-3)(5-s-D)+4]\ \nabla\cdot\alpha\right\}\,.
\end{aligned} \tag{2.37}$$

Current conservation is guaranteed in this case by the AdS generalization of eq. (2.21),

$$\nabla\cdot E_{\varphi\,L}(k)\ =\ \frac{1}{3\,\binom{s}{3}}\,g\,E_{\alpha\,L}(k)\ -\ \frac{1}{4\,\binom{s}{4}}\,\nabla^{\,3}\,E_{\beta\,L}(k)\,, \tag{2.38}$$

a result that reflects the Noether identity signalling the gauge invariance of the action, which indeed implies that

$$\int d^Dx\ \left[-\ s\ \Lambda\ \nabla\ \cdot\ \frac{\delta\mathcal{L}_k}{\delta\varphi}\ +\ \nabla\cdot\nabla\cdot\nabla\cdot\Lambda\ \frac{\delta\mathcal{L}_k}{\delta\beta}\ +\ \Lambda\,'\,\frac{\delta\mathcal{L}_k}{\delta\alpha}\right]\ =\ 0\ . \tag{2.39}$$

2.2 Fermions

The fermionic case is more involved but does not add substantial novelties with respect to what we have seen for bosons. As in the preceding Subsection, we can now begin by simplifying and generalizing the construction of gauge invariant Lagrangians for unconstrained free fields of [18]. In doing so, we shall be able to stress the key role of a few gauge-invariant blocks, a step that will prove again quite convenient when constructing the AdS deformation of the flat-space results. Let us therefore begin by recalling the definition of the Fang-Fronsdal operator for a totally symmetric rank-n tensor-spinor ψ [1],

$$\mathcal{S}\ =\ i\ (\not\partial\,\psi\ -\ \partial\ \not\psi)\ , \tag{2.40}$$

where $\not\psi$ denotes the γ-trace of the gauge field. Under the gauge transformation

$$\delta\,\psi\ =\ \partial\,\epsilon\,, \tag{2.41}$$

$\mathcal{S}$ varies according to

$$\delta\,\mathcal{S} \,=\, -\,2\,i\,\partial^2\,\not{\epsilon}\,, \tag{2.42}$$

where $\not{\epsilon}$ denotes the γ-trace of the gauge parameter ϵ. In analogy with what we have seen for bosons, one can build from $\mathcal{S}$ the *fully* gauge invariant operator

$$\mathcal{W} \,\equiv\, S \,+\, 2\,i\,\partial^2\xi\,, \tag{2.43}$$

where the rank-$(n-2)$ *compensator* ξ transforms as

$$\delta\,\xi \,=\not{\epsilon} \tag{2.44}$$

under the gauge transformation (2.41).

The Bianchi identity for $\mathcal{W}$,

$$\partial\cdot\mathcal{W} \,-\, \frac{1}{2}\,\partial\,\mathcal{W}' \,-\, \frac{1}{2}\,\not{\partial}\,\not{\mathcal{W}} \,=\, i\,\partial^2\left[\not{\psi}' \,-\, 2\,\partial\cdot\xi \,-\, \partial\,\xi' -\, \not{\partial}\,\not{\xi}\,\right]\,, \tag{2.45}$$

leads naturally to a second gauge-invariant tensor-spinor,

$$\mathcal{Z} \,\equiv\, i\left\{\not{\psi}' \,-\, 2\,\partial\cdot\xi \,-\, \partial\,\xi' -\, \not{\partial}\,\not{\xi}\right\}\,, \tag{2.46}$$

directly related to the triple γ-trace constraint on the fermionic gauge field ψ, that is absent in the Fang-Fronsdal formulation. The minimal flat-space Lagrangians of [18] can then be recovered starting from the trial Lagrangians

$$\mathcal{L}_0 \,=\, \frac{1}{2}\,\bar{\psi}\,\left(\mathcal{W} \,-\, \frac{1}{2}\,\eta\,\mathcal{W}' \,-\, \frac{1}{2}\,\gamma\,\not{\mathcal{W}}\right) \,+\, h.c.\,, \tag{2.47}$$

and compensating the remainders in their gauge transformations with new terms involving the field ξ and the tensor $\mathcal{Z}$. The complete Lagrangians are finally

$$\begin{aligned}\mathcal{L} =&\frac{1}{2}\,\bar{\psi}\,\left(\mathcal{W} \,-\, \frac{1}{2}\,\gamma\not{\mathcal{W}} \,-\, \frac{1}{2}\,\eta\,\mathcal{W}'\right) \,-\, \frac{3}{4}\binom{n}{3}\,\bar{\not{\xi}}\,\partial\cdot\mathcal{W}' \\ &+\, \frac{1}{2}\binom{n}{2}\,\bar{\xi}\,\partial\cdot\not{\mathcal{W}} \,+\, \frac{3}{2}\binom{n}{3}\bar{\lambda}\,\mathcal{Z} \,+\, h.c.\,,\end{aligned} \tag{2.48}$$

where, as in [18], we have introduced a *Lagrange multiplier* λ, whose gauge transformation is determined to be

$$\delta\lambda \,=\, \partial\cdot\partial\cdot\epsilon \tag{2.49}$$

in order for $\mathcal{L}$ to be gauge invariant. As in the bosonic case, it is useful to introduce an additional gauge-invariant tensor, now involving λ,

$$\mathcal{Y} \,\equiv\, i\,\left[\lambda \,-\, \frac{1}{2}\,(\partial\cdot\psi' \,-\, \Box\,\not{\xi} \,-\, \partial\,\partial\cdot\,\not{\xi})\right]\,. \tag{2.50}$$

since only this gauge-invariant combination can actually enter the field equations.

The identification of the $\mathcal{Y}$ tensor suggests again that (2.48) is but a member of a family of Lagrangians depending on a parameter k,

$$\begin{aligned}\mathcal{L} \,=&\frac{1}{2}\,\bar{\psi}\,\left(\mathcal{W} \,-\, \frac{1}{2}\,\eta\,\mathcal{W}' \,-\, \frac{1}{2}\,\gamma\not{\mathcal{W}}\right) \,-\, \frac{3}{4}\binom{n}{3}\,\bar{\not{\xi}}\,\partial\cdot\mathcal{W}' \\ &+\, \frac{1}{2}\binom{n}{2}\,\bar{\xi}\,\partial\cdot\not{\mathcal{W}} \,+\, \frac{3}{2}\binom{n}{3}\,(\bar{\lambda} \,-\, i\,k\,\bar{\mathcal{Y}})\,\mathcal{Z} \,+\, h.c.\,,\end{aligned} \tag{2.51}$$

where the previous case of eq. (2.48) corresponds to the choice $k = 0$. Other possible quadratic terms in $\mathcal{Z}$ are excluded as irrelevant, for the same reasons as in the bosonic case. The field equations following from (2.51) are then

$$
\begin{aligned}
E_{\bar{\psi}}(k) &\equiv \mathcal{W} - \frac{1}{2}\gamma \not{\mathcal{W}} - \frac{1}{2}\eta\,\mathcal{W}' - \frac{1}{4}(1+k)\,\eta\,\partial\,\mathcal{Z} - \frac{1}{2}(1-k)\,\eta\,\gamma\,\mathcal{Y} = 0\,, \\
E_{\bar{\xi}}(k) &\equiv \binom{n}{2}\Big\{\partial\cdot\not{\mathcal{W}} + \frac{1}{2}\gamma\,\partial\cdot\mathcal{W}' + (1-k)\big(\partial + \frac{1}{2}\gamma\,\not{\partial} + \eta\,\partial\cdot\big)\,\mathcal{Y} \\
&\qquad + \frac{1+k}{4}\gamma\,(\Box + \partial\,\partial\cdot)\,\mathcal{Z}\Big\} = 0\,, \\
E_{\bar{\lambda}}(k) &\equiv \frac{3}{2}\binom{n}{3}(1-k)\,\mathcal{Z} = 0\,.
\end{aligned}
\tag{2.52}
$$

As in the bosonic case, they can be reduced in general to the compensator equations of [13, 16]

$$
\begin{aligned}
\mathcal{W} &\equiv S + 2\,i\,\partial^2\,\xi = 0\,, \\
\mathcal{Z} &\equiv i\,\{\not{\psi}' - 2\,\partial\cdot\xi - \partial\,\xi' - \not{\partial}\,\not{\xi}\} = 0\,,
\end{aligned}
\tag{2.53}
$$

and eventually to the Fang-Fronsdal form upon partial gauge fixing.

In complete analogy with the bosonic case, when $\bar{\psi}$ is coupled to a *conserved* external source the consistency of the system is guaranteed by the field equations for the two auxiliary fields $\bar{\xi}$ and $\bar{\lambda}$, since

$$
\partial\cdot E_{\bar{\psi}}(k) = \frac{1}{3\binom{n}{3}}\,\partial^2\,E_{\bar{\lambda}}(k) - \frac{1}{2\binom{n}{2}}\,\gamma\,E_{\bar{\xi}}(k)\,. \tag{2.54}
$$

Moreover, in terms of the tensors defined in (2.52), the Lagrangian finally takes the simple form

$$
\mathcal{L}_k = \frac{1}{2}\,\bar{\psi}\,E_{\bar{\psi}}(k) + \frac{1}{2}\,\bar{\xi}\,E_{\bar{\xi}}(k) + \frac{1}{2}\,\bar{\lambda}\,E_{\bar{\lambda}}(k) + h.c.\,. \tag{2.55}
$$

Generalizing these expressions to an *AdS* background entails a few complications. First of all, the gauge transformation of ψ acquires an additional contribution [31], a phenomenon that has a well-known counterpart in gauged supergravity, so that now

$$
\delta\psi = \nabla\epsilon + \frac{1}{2\,L}\,\gamma\,\epsilon\,. \tag{2.56}
$$

Hence, with our "mostly plus" convention for the space-time signature,

$$
\delta\bar{\psi} = \nabla\bar{\epsilon} - \frac{1}{2\,L}\,\bar{\epsilon}\,\gamma\,. \tag{2.57}
$$

The fermionic gauge transformation lends naturally to define a modified covariant derivative,

$$
\Delta \equiv \nabla + \frac{1}{2\,L}\gamma\,, \tag{2.58}
$$

in terms of which the basic relations take a simpler form. The starting point is again the covariantized Fang-Fronsdal operator, that in an *AdS* background is also modified by the addition of mass-like terms and reads

$$
\mathcal{S}_L = i\,(\not{\Delta}\,\psi - \Delta\,\not{\psi}) + \frac{i}{L}\,(n-2)\,\psi + \frac{i}{L}\gamma\,\not{\psi}\,, \tag{2.59}
$$

One can now show that $\mathcal{S}_L$ transforms as

$$\delta\,\mathcal{S}_L \,=\, -\,i\,\left(\Delta^{\,2}\,\not{\epsilon}\,-\,\frac{2}{L}\,\gamma\,\Delta\,\not{\epsilon}\right)\,, \tag{2.60}$$

under the gauge transformation and satisfies the Bianchi identity

$$\Delta\cdot\mathcal{S}_L\,-\,\frac{1}{2}\,\not{\Delta}\,\not{\mathcal{S}}_L\,-\,\frac{1}{2}\,\Delta\,\mathcal{S}_L^{\,\prime}\,=\,\frac{1}{2\,L}\,(n\,-\,1)\,\not{\mathcal{S}}_L\,-\,\frac{1}{2\,L}\,\gamma\,\mathcal{S}_L^{\,\prime}\,+\,\frac{i}{2}\,\left(2\,\Delta^{\,2}\,-\,\frac{2}{L}\,\gamma\,\Delta\right)\,\not{\psi}^{\,\prime}\,, \tag{2.61}$$

a simpler expression than the corresponding one in terms of the more conventional derivative ∇,

$$\begin{aligned}\nabla\cdot\mathcal{S}_L\,-\,\frac{1}{2}\,\not{\nabla}\,\not{\mathcal{S}}_L\,-\,\frac{1}{2}\,\nabla\,\mathcal{S}_L^{\,\prime}\,=&-\,\frac{1}{4L}\,\gamma\,\mathcal{S}_L^{\,\prime}\,+\,\frac{1}{4L}\,[(D\,-\,2)\,+\,2\,(n\,-\,1)]\,\not{\mathcal{S}}_L\\&+\,\frac{i}{2}\,\left(2\,\nabla^2\,-\,\frac{1}{L}\,\gamma\,\nabla\,-\,\frac{3}{2\,L^2}\,g\right)\,\not{\psi}^{\,\prime}\,,\end{aligned} \tag{2.62}$$

where we are correcting a couple of misprints present in eq. (5.37) of [16]. One may notice, in particular, that the space-time dimension D never appears explicitly in eq. (2.61).

As in the bosonic case, one is led to define the deformed gauge-invariant structures

$$\begin{aligned}\mathcal{W}_L\,&=\,\mathcal{S}_L\,+\,2\,i\,\Delta^{\,2}\,\xi\,-\,\frac{2\,i}{L}\,\gamma\,\Delta\,\xi\,,\\\mathcal{Y}_L\,&=i\,\left\{\lambda-\frac{1}{2}(\Delta\cdot\not{\psi}^{\,\prime}-\Box\,\not{\xi}-\Delta\,\Delta\cdot\,\not{\xi}\,-\,\frac{1}{L^2}[(n-3)(5-n-D)\,\not{\xi}\,+\,2\,g\,\not{\xi}^{\,\prime}]\right\}\,,\\\mathcal{Z}_L\,&=i\,\left\{\not{\psi}^{\,\prime}\,-\,2\,\Delta\cdot\xi\,-(\,\Delta\,-\,\frac{1}{L}\,\gamma)\,\xi^{\,\prime}\,-\,(\not{\Delta}\,+\,\frac{1}{L}\,(n\,-\,3))\,\not{\xi}\right\}\,,\end{aligned} \tag{2.63}$$

where in particular the $\mathcal{Y}_L$ tensor is gauge-invariant provided the field λ transforms according to

$$\delta\,\lambda\,=\,\Delta\cdot\Delta\cdot\epsilon\,. \tag{2.64}$$

The Bianchi identity satisfied by $\mathcal{W}_L$,

$$\Delta\cdot\mathcal{W}_L-\frac{1}{2}\,\not{\Delta}\,\not{\mathcal{W}}_L-\frac{1}{2}\,\Delta\,\mathcal{W}_L^{\,\prime}\,=\,\frac{1}{2\,L}\,(n-1)\,\not{\mathcal{W}}_L-\frac{1}{2\,L}\,\gamma\,\mathcal{W}_L^{\,\prime}+\frac{1}{2}\,\left(2\,\Delta^{\,2}\,-\,\frac{2}{L}\,\gamma\,\Delta\right)\,\mathcal{Z}_L\,, \tag{2.65}$$

or, what is equivalent, the divergence of the kinetic operator constructed from it,

$$\Delta\cdot\left\{\mathcal{W}_L\,-\,\frac{1}{2}\,\gamma\,\not{\mathcal{W}}_L\,-\,\frac{1}{2}\,g\,\mathcal{W}_L^{\,\prime}\right\}\,=\,-\,\frac{1}{2}\,\gamma\,\Delta\cdot\,\not{\mathcal{W}}_L\,-\,\frac{1}{2}\,g\,\Delta\,\cdot\,\mathcal{W}_L^{\,\prime}\,+\,\frac{1}{2}\,\left(2\,\Delta^{\,2}\,-\,\frac{2}{L}\,\gamma\,\Delta\right)\,\mathcal{Z}_L\,, \tag{2.66}$$

is then the main ingredient to build the one-parameter family of gauge-invariant Lagrangians

$$\begin{aligned}\mathcal{L}\,=&\,\frac{e}{2}\,\bar{\psi}\,\left(\mathcal{W}_L\,-\,\frac{1}{2}\,\gamma\,\not{\mathcal{W}}_L\,-\,\frac{1}{2}\,g\,\mathcal{W}_L^{\,\prime}\right)\,-\,\frac{3e}{4}\,\binom{n}{3}\,\bar{\not{\xi}}\,\Delta\cdot\mathcal{W}_L^{\,\prime}\\&+\,\frac{e}{2}\,\binom{n}{2}\,\bar{\xi}\,\Delta\cdot\not{\mathcal{W}}_L\,+\,\frac{3\,e}{2}\,\binom{n}{3}\,\left(\bar{\lambda}\,-\,\frac{2}{L}\,\Delta\cdot\bar{\xi}\,-\,i\,k\,\mathcal{Y}_L\right)\,\mathcal{Z}_L\,+\,h.c.\,.\end{aligned} \tag{2.67}$$

From these one can finally derive the AdS field equations,

$$
\begin{aligned}
E_{\bar{\psi}\,L}(k) &\equiv \mathcal{W}_L \,-\, \frac{1}{2}\gamma \not{\mathcal{W}}_L \,-\, \frac{1}{2} g \mathcal{W}_L^{\prime} \,-\, \frac{1}{4}(1\,+\,k)\, g\, \Delta\, \mathcal{Z}_L \,-\, \frac{1}{2}(1\,-\,k)\,\gamma\, g\, \mathcal{Y}_L \,=\, 0\,, \\
E_{\bar{\xi}\,L}(k) &\equiv \binom{n}{2} \Big\{ \Delta \cdot \not{\mathcal{W}}_L \,+\, \frac{1}{2}\gamma\, \Delta \cdot\, \mathcal{W}_L^{\prime} + (1\,-\,k) \left(\Delta\, \mathcal{Y}_L \,+\, \frac{1}{L}\Delta \cdot\, (\gamma\, \mathcal{Y}_L) \right) \\
&\qquad + \frac{2}{L}\, \Delta\, \mathcal{Z}_L \,+\, \frac{1\,+\,k}{4}\, \gamma\, \Delta \cdot (\Delta\, \mathcal{Z}_L) \Big\} \,=\, 0\,, \\
E_{\bar{\lambda}\,L}(k) &\equiv \frac{3}{2}\binom{n}{3}(1\,-\,k)\, \mathcal{Z}_L \,=\, 0\,.
\end{aligned}
\tag{2.68}
$$

The conservation on an external current coupled to ψ is finally implied by the analog of (2.54),

$$
\Delta \cdot E_{\bar{\psi}\,L}(k) \,=\, \frac{1}{6\binom{n}{3}}\, \Delta^{\,2}\, E_{\bar{\lambda}\,L}(k) \,-\, \frac{1}{2\binom{n}{2}}\, \gamma\, E_{\bar{\xi}\,L}(k)\,. \tag{2.69}
$$

3 Current exchanges in the local formulations

In this Section we consider the response of unconstrained higher-spin gauge fields to external currents, focussing in particular on the current exchange. This is a convenient device to compare different formulations in the simplest possible setting, and the ensuing analysis is indeed rather rewarding. As we shall see, the constrained Fronsdal formulation agrees directly, in this respect, with the minimal unconstrained form discussed in the previous Section, but not with the non-local geometric formulation of [7, 13]. The lack of direct agreement between the source couplings in the local and non-local forms of the theory, however, is an interesting fact that can be turned to our own advantage: it determines a unique form for the non-local theory, selecting a specific form for the corresponding Lagrangian.

In the absence of sources, as stressed in [13], the iterative procedure of [7] builds a sequence of pseudo-differential operators

$$
\mathcal{F}^{(n+1)} \;=\; \mathcal{F}^{(n)} \;+\; \frac{1}{(n+1)(2n+1)}\, \frac{\partial^{\,2}}{\Box}\, \mathcal{F}^{(n)\,\prime} \;-\; \frac{1}{n+1}\, \frac{\partial}{\Box}\, \partial\cdot \mathcal{F}^{(n)}\;, \tag{3.1}
$$

turning the first of the compensator equations (2.19) into a sequence of non-local equations,

$$
\mathcal{F}^{(n)} \;=\; (2\,n\;+\;1)\, \frac{\partial^{\,2n+1}}{\Box^{n-1}}\, \alpha^{[n-1]}\;. \tag{3.2}
$$

Eventually one thus arrives at an irreducible form involving the gauge field φ alone, that is to be expressible in terms of higher-spin curvatures. This gauge invariant form, however, is clearly not unique. For instance, after α has disappeared, further iterations of (3.1) will produce additional gauge invariant equations. However, as we shall see in the next Section, external sources select a unique form of the non-local equations, that should be regarded as the proper counterpart of the Lagrangian equations of the previous Section, although they will be somewhat more complicated than the Lagrangian equations of [7]. We can now begin to investigate these issues in some detail for the local formulation, in order to show the equivalence of its constrained and unconstrained forms.

3.1 Bose fields in flat space

In order to motivate our procedure, let us begin by rephrasing a simple and familiar example, the comparison between the light-cone and covariant forms of the Maxwell theory. In the former, one has the $D-2$ transverse components A_i of the gauge field, that in momentum space couple to an external current j_i according to

$$p^2 A_i \;=\; j_i \; . \tag{3.3}$$

The current-current interaction in this physical gauge is thus sized by the product of two purely transverse currents, since

$$p^2 j_i A_i \;=\; j_i \, j_i \; . \tag{3.4}$$

In a similar fashion, in the covariant formulation one would start from the Maxwell equation for the full vector potential,

$$\left(p^2 \eta_{\mu\nu} \;-\; p_\mu p_\nu\right) A^\nu \;=\; J_\mu \; , \tag{3.5}$$

where, however, consistency demands that the current be *conserved*. As a result, in this case the current-current interaction is sized by the full scalar product of a pair of covariant *conserved* currents, since

$$p^2 J^\mu A_\mu \;=\; J^\mu J_\mu \; . \tag{3.6}$$

Incidentally, here one faces a cute, albeit well-known, fact: folding the kinetic operator into a conserved current has made it possible to effectively recover the propagator without a gauge-fixing procedure. The issue is now to proceed to the (singular) on-shell limit, in order to compare the number of degrees of freedom exchanged in the two formulations. These are encoded in the Euclidean product $j_i \, j_i$ for eq. (3.4) and in the Lorentzian product $J_\mu \, J^\mu$ for eq. (3.6), which, differently from the propagators, are well defined on-shell.

An arbitrary on-shell current $J_\mu(p)$ can be made transverse upon multiplication by the projector

$$\Pi_{\mu\nu} \;=\; \eta_{\mu\nu} \;-\; p_\mu \bar{p}_\nu \;-\; p_\nu \bar{p}_\mu \; , \tag{3.7}$$

where p is the exchanged on-shell momentum, so that $p^2 = 0$, and $\bar{p}$ is a second vector such that $\bar{p}^2 = 0$ and $p \cdot \bar{p} = 1$. Notice that the projector Π satisfies the conditions

$$\Pi_{\mu\nu} \, \Pi^{\nu\rho} \;=\; \Pi_\mu{}^\rho \; , \quad \eta_{\mu\nu} \, \Pi^{\mu\nu} \;=\; D-2 \; , \quad \Pi_{\mu\nu} \, p^\nu \;=\; 0 \; . \tag{3.8}$$

The same number of polarizations is thus exchanged in both cases: in the former one has directly $(D-2)$ of them in D space-time dimensions, while the latter involves $J^\mu \, \Pi_{\mu\nu} \, J^\nu$, so that the trace of the projector $\Pi_{\mu\nu}$, again equal to $D-2$, leads to the same result even starting with full D-dimensional covariant currents.

One can repeat almost verbatim the exercise for a spin-2 field. For all dimensions $D \geq 3$, its degrees of freedom in the light-cone exchange fill the $\frac{D(D-3)}{2}$ independent components a symmetric traceless tensor in $(D-2)$ dimensions, while now the covariant Lagrangian equation is

$$p^2 h_{\mu\nu} \;-\; p_\mu \, p \cdot h_\nu \;-\; p_\nu \, p \cdot h_\mu \;+\; p_\mu \, p_\nu \, h^{\,\prime} \;-\; \eta_{\mu\nu} \left(p^2 \, h^{\,\prime} \;-\; p \cdot p \cdot h\right) \;=\; J_{\mu\nu}(p) \; . \tag{3.9}$$

Combining this equation with its trace then gives

$$p^2 h_{\mu\nu} \;-\; p_\mu \, p \cdot h_\nu \;-\; p_\nu \, p \cdot h_\mu \;+\; p_\mu \, p_\nu \, h^{\,\prime} \;=\; J_{\mu\nu} \;-\; \frac{\eta_{\mu\nu}}{D-2} \, J^{\,\prime} \tag{3.10}$$

and finally folding it, as above, with the conserved current $J_{\mu\nu}(p)$,

$$p^2 J^{\mu\nu}\, h_{\mu\nu} \;=\; J^{\mu\nu}\, J_{\mu\nu} \;-\; \frac{1}{D-2}\; \left(J\,'\right)^2 \;, \tag{3.11}$$

or equivalently

$$p^2 J^{\mu\nu}\, h_{\mu\nu} \;=\; \left(J_{\mu\nu} \;-\; \frac{1}{D-2}\, \Pi_{\mu\nu}\, J\,'\right)^2 \;, \tag{3.12}$$

where the expression within brackets is the traceless and transverse projection of J. In deriving this result we have taken into account that, as in the spin-one case, the conservation of $J_{\mu\nu}$ forces the projector Π into the expression. The end conclusion is, therefore, that the degrees of freedom interchanged in the covariant formulation fill a traceless symmetric matrix in $D-2$ dimensions, just like their light-cone counterparts.

The previous discussion gives us the flavor of the general case, although it inevitably leaves out its key subtleties. Thus, in D dimensions the degrees of freedom carried by a massless fully symmetric tensor $\varphi_{\mu_1\ldots\mu_s}$ should fill an irreducible representation of the little group $SO(D-2)$ corresponding to a *traceless* symmetric tensor $j_{a_1\ldots a_s}$. Standard properties of Young tableaux determine the dimension of a traceful rank-s symmetric tensor of $SO(D)$,

$$P(D,s) = \frac{(D+s-1)!}{(D-1)!\, s!} \;, \tag{3.13}$$

while for a traceless symmetric tensor of the same rank in $D-2$ dimensions the corresponding number is

$$P(D-2,s) \;-\; P(D-2,s-2) \;=\; \frac{(D+2s-4)(D+s-5)!}{(D-4)!\, s!} \;. \tag{3.14}$$

Only for $s = 1, 2$, however, can this result be also expressed as $P(D,s)-2P(D,s-1)$, the relation suggested by the two simple examples above. In both these cases, one can indeed first subtract $P(D,s-1)$ degrees of freedom to account for the Lorentz (or de Donder) gauge conditions, one for $s=1$ and a full vector for $s=2$, and then again the same number, $P(D,s-1)$, to account for a second, on-shell, gauge transformation that preserves the first. This is at the origin of the simple description of the $s=1$ (Maxwell) and $s=2$ (Einstein) cases by unconstrained fields with a local gauge invariance, while the discrepancy for $s>2$ reflects the novel features of higher-spin fields, the need for Fronsdal's trace conditions as in [1] or for the compensators of the previous Section.

Returning to our main task we would like to discuss, in a Lorentz-covariant formalism and for an arbitrary tensor $\varphi_{\mu_1\ldots\mu_s}$, the current exchange for a pair of totally symmetric currents $J_{\mu_1\ldots\mu_s}$. As anticipated, the light-cone construction forces the result to equal $j_{a_1\ldots a_s}\, j^{a_1\ldots a_s}$ on shell, and the issue at stake is whether, in the various available formulations, the basic equality

$$J_{\mu_1\ldots\mu_s}\; \mathcal{P}^{\mu_1\ldots\mu_s;\nu_1\ldots\nu_s}\; J_{\nu_1\ldots\nu_s} \;=\; j_{a_1\ldots a_s}\; j^{a_1\ldots a_s} \;, \tag{3.15}$$

holds, with $\mathcal{P}$ the spin-s analogue of the tensor defined by the *r.h.s.* of eq. (3.10). In the following we shall describe how this identity, that guarantees that only physical degrees of freedom are exchanged, can be recovered in the local, constrained or unconstrained, formulations of higher-spin gauge fields. In Fronsdal's construction, where both the double trace of the current J and the traceless part of its divergence vanish, this result was presented in [1], and the derivation is repeated here for completeness, while simply extending it to D dimensions.

We can now build the *l.h.s.* of eq. (3.15). To this end, let us begin by noticing that, given a generic totally symmetric current J, its projection onto a traceless symmetric tensor can be attained via the sum

$$T_s J \; = \; \sum_0^N \rho_n(D,s)\, \eta^n \, J^{[n]}, \tag{3.16}$$

where $J^{[n]}$ denotes the n^{th} trace of J and N is the smallest integer such that the next trace $J^{[N+1]}$ can not be defined. The coefficients of this expansion depend on the spin s of J and on the space-time dimension D. They are determined by the results collected in the Appendix, that lead to the one-term recursion relation

$$\rho_{n+1}(D,s) \; = \; - \, \frac{\rho_n(D,s)}{D+2(s-n-2)} \ , \tag{3.17}$$

with the initial condition $\rho_0(D,s)=1$. Notice that $\rho_n(D-2,s)=\rho_n(D,s-1)$.

As we have seen, a generic tensor can be rendered transverse by the application of the projector Π. In order to build the tensor $\mathcal{P}$ that defines the current exchange, one can first project onto the transverse part of J to then extract the traceless part of the resulting tensor, thus obtaining

$$\mathcal{P}\; J \; = \; \sum_{n=0}^N \rho_n(D-2,s)\, \Pi^{\; n} \, J^{[n]} \ . \tag{3.18}$$

This is just what we have seen in eq. (3.16), but for a key novelty: here ρ depends on $D-2$, as a result of the presence of Π.

We have thus made our way backwards to the tensor $\mathcal{P}$ of eq. (3.15), that projects a covariant current J onto its transverse traceless part, but at the price of an explicit dependence on both the physical momentum p and the additional vector $\bar{p}$. In some notable cases, however, the dependence on $\bar{p}$ disappears. For instance, if J is conserved,

$$J \cdot \mathcal{P} \cdot J \; = \; \sum_{n=0}^N \rho_n(D-2,s)\, \frac{s!}{n!\,(s-2n)!\,2^n} \; J^{[n]} \cdot J^{[n]}, \tag{3.19}$$

so that $\mathcal{P}$ can be effectively replaced by

$$\mathcal{P}_c J \; = \; \sum_{n=0}^N \rho_n(D-2,s)\, \eta^n \, J^{[n]} \ , \tag{3.20}$$

and one can now show that

$$\begin{aligned} (\mathcal{P}_c J)' \; &= \; 2 \sum_0^{N-1} \rho_{n+1}(D-2,s)\, \eta^{\,n} \, J^{\,[n+1]} \, , \\ (\mathcal{P}_c J)'' \; &= \; 0 \, . \end{aligned} \tag{3.21}$$

The momentum dependence of $\mathcal{P}$ can be also suppressed in Fronsdal's constrained formulation [1]. The Lagrangian equation is in this case

$$\mathcal{F} \; - \; \frac{1}{2}\, \eta \, \mathcal{F}^{\,\prime} \; = \; J \ , \tag{3.22}$$

and the double-trace condition may be used to show that

$$J' \ = \ - \ \frac{1}{2\,\rho_1(D-2,s)}\ \mathcal{F}' \ , \tag{3.23}$$

while the double trace of J vanishes,

$$J'' \ = \ 0 \ , \tag{3.24}$$

so that eq. (3.22) can be also presented in the alternative form

$$\mathcal{F} \ = \ J \ + \ \rho_1(D-2,s)\,\eta\,J' \ . \tag{3.25}$$

In addition, eq. (3.22) implies that

$$p \cdot J \ = \ - \ \frac{1}{2}\,\eta\,p \cdot \mathcal{F}' \ , \tag{3.26}$$

so that, using eq. (3.23), one can see that *only* the traceless part of the divergence of J vanishes in general. The last condition can be also written

$$p \cdot J \ + \ \rho_1(D,s-1)\ \eta\ p \cdot J' \ = \ 0 \ , \tag{3.27}$$

so that, introducing a new tensor τ, the combination

$$J \ + \rho_1(D-2,s)\,\eta\,J' \ + \ p\,\tau \tag{3.28}$$

is actually both *traceless* and *transverse* provided τ is *traceless* and satisfies the condition

$$p \cdot \tau \ = \ - \ \rho_1(D-2,s)\ J' \ . \tag{3.29}$$

However, $p\,\tau$ yields a vanishing result upon contraction with J since, as we have seen, the traceless part of $p \cdot J$ vanishes. The end conclusion is that in Fronsdal's constrained theory $\mathcal{P}J$ can be actually replaced with

$$\mathcal{P}_c\ J \ = \ J \ + \ \rho_1(D,s-1)\ \eta\ J' \ , \tag{3.30}$$

and therefore the corresponding current exchange is determined by the relatively simple expression

$$J \cdot J \ + \ \frac{\rho_1(D-2,s)\,s\,(s-1)}{2}\ J' \cdot J' \ . \tag{3.31}$$

In order to compare with the unconstrained formulation of the previous Section, let us begin by noticing that, once the third of eqs. (2.17), the constraint $\mathcal{C}=0$, is enforced, the coupling to an external source is described by

$$\mathcal{A} \ - \ \frac{1}{2}\,\eta\,\mathcal{A}' \ + \eta^2\ \mathcal{B} = \ J \tag{3.32}$$

while the double trace $\mathcal{A}''$ vanishes identically. As a result, the quantity

$$K = J \ - \ \eta^2\,\mathcal{B} \tag{3.33}$$

is somehow the counterpart of Fronsdal's current in this case, since on shell $K''=0$ as a result of the condition $\mathcal{A}''=0$, that as we have stated follows from the constraint $\mathcal{C}=0$. This condition also determines $\mathcal{B}$, and thus the Lagrange multiplier β, in terms of J.

One can now write

$$K \;=\; J \;+\; \sum_{n=2}^{N} \sigma_n \, \eta^n \, J^{[n]}, \tag{3.34}$$

with the coefficients determined recursively by the condition that the double trace $K\,''$ vanish. This apparently results in a three-term recursion relation,

$$\sigma_n \;+\; \big[D+2(s-n-3)\big] \left\{ 2\sigma_{n+1} \;+\; \big[D+2(s-n-4)\big]\,\sigma_{n+2} \right\} \;=\; 0 \;, \tag{3.35}$$

with the conditions

$$\sigma_2\,\big[D+2(s-3)\big]\,\big[D+2(s-4)\big] \;=\; -\,1, \;\; 2\,\sigma_2 \;+\; \sigma_3\,\big[D+2(s-5)\big] \;=\; 0 \;. \tag{3.36}$$

However, introducing

$$u_n \;=\; \sigma_n \;+\; \big[D+2(s-n-3)\big]\,\sigma_{n+1} \;, \tag{3.37}$$

one can turn eq. (3.35) into the simpler two-term recursion relation,

$$u_{n+1} \;=\; -\; u_n \, \frac{1}{\big[D+2(s-n-3)\big]} \;, \tag{3.38}$$

which, taking into account the value of u_2 determined by eqs. (3.36),

$$u_2 \;=\; \frac{1}{\big[D+2(s-3)\big]\,\big[D+2(s-4)\big]} \;, \tag{3.39}$$

is actually solved by

$$u_n = \rho_n(D-2,s) \;, \tag{3.40}$$

where the ρ_n are defined in eq. (3.17).

Making use of (3.38), the defining relation for the u_n becomes

$$u_n \;=\; \sigma_n \;-\; \frac{u_n}{u_{n+1}}\,\sigma_{n+1} \;, \tag{3.41}$$

or, in terms of $v_n = \sigma_n/u_n$,

$$v_{n+1} \;-\; v_n + 1 \;=\; 0 \;, \tag{3.42}$$

whose solution is $v_n = v_2 - (n-2) = -n+1$. In conclusion,

$$\sigma_n \;=\; (-n+1)\;\rho_n(D-2,s) \;. \tag{3.43}$$

Since $\mathcal{A}$ is doubly traceless, eq. (3.32) can be turned into

$$\mathcal{A} \;=\; K \;+\; \rho_1(D-2,s)\,\eta\,K\,' \;, \tag{3.44}$$

and one can verify that

$$K \;+\; \rho_1(D-2,s)\,\eta\,K\,' \;=\; \sum_n \; \rho_n(D-2,s)\;\eta^n\;J^{[n]} \;. \tag{3.45}$$

This determines $\mathcal{P}\,J$, and the current exchange is finally

$$\sum_{n=0}^{N} \; \rho_n(D-2,s)\;\frac{s!}{n!\;(s-2n)!\;2^n}\;J^{[n]}\cdot J^{[n]} \tag{3.46}$$

which agrees with eq. (3.19), with the correct number of degrees of freedom, since the exchange involves, as expected, a pair of traceless conserved currents built from the original conserved current J.

3.2 Fermi fields in flat space

In the previous Subsection we have recalled how, in D dimensions, the physical degrees of freedom carried by a massless symmetric rank-s tensor $\varphi_{\mu_1\ldots\mu_s}$ fill a symmetric *traceless* rank-s tensor in $D-2$ transverse dimensions. For massless Fermi fields the situation is similar, if technically more involved, so that a D-dimensional spinor-tensor field $\psi_{\mu_1\ldots\mu_m}$ carries physical degrees of freedom corresponding to an *on-shell* γ*-traceless* spinor-tensor in $D-2$ transverse dimensions. The on-shell condition is of course stronger for a Fermi field: not only does it select a light-like momentum, but via the Dirac equation it further halves the number of its degrees of freedom. This is reflected in the familiar presence, in spinor propagators, of the matrix $\not{p}$, that for a light-like momentum p has precisely this effect.

In discussing current exchanges for Fermi fields, it is useful to begin by defining the projection of $\psi_{\mu_1\ldots\mu_m}$ to its γ-traceless part. One can verify that this is effected by

$$\mathcal{T}_m\psi \;=\; \psi \;+\; \sum_{n=1}^{N} \rho_n(D,m+1)\left(\eta^n\psi^{[n]} \;+\; \eta^{n-1}\gamma\;\not{\psi}^{[n-1]}\right), \tag{3.47}$$

where the coefficients $\rho_n(D,s)$ were introduced in the previous Subsection, since they also determine the traceless projection for bosonic fields.

Using $\mathcal{T}_m$, one can also construct the projectors to the doubly and triply γ-traceless parts of ψ. The first is simply the traceless projector that was introduced in the previous Subsection, but here we can also relate it to the projector of eq. (3.47), according to

$$\mathcal{T}^{(2)}_m\psi \;=\; \sum_n \rho_n(D,m)\,\eta^n\,\psi^{[n]} \;=\; \mathcal{T}_m\,\psi \;+\; \frac{1}{D+2m-2}\,\gamma\,\mathcal{T}_{m-1}\;\not{\psi}\;. \tag{3.48}$$

In a similar fashion, one can define a triply γ-traceless projector,

$$\begin{aligned}\mathcal{T}^{(3)}_m\psi \;&=\; \mathcal{T}^{(2)}_m\,\psi \;+\; \frac{1}{D+2m-4}\,\eta\,\mathcal{T}_{m-2}\,\psi' \\ &=\; \psi \;-\; \eta\,\gamma\left(\frac{1}{(D+2m-6)(D+2m-4)}\,\not{\psi}'+\ldots\right).\end{aligned} \tag{3.49}$$

In deriving these results, use has been made of the relations

$$\rho_n(D-2,m+1) \;=\; \rho_n(D,m)\;, \quad \rho_n(D,m-1) \;=\; \frac{\rho_{n+1}(D,m)}{\rho_1(D,m)}\;, \tag{3.50}$$

which follow from eq. (3.17), and of the initial condition $\rho_0(D,s)=1$.

If the propagator is now denoted by $\frac{\not{p}}{p^2}\,\mathcal{B}$, the previous considerations lead to

$$\mathcal{B}\,\mathcal{J} \;=\; \mathcal{J} \;+\; \sum_{n=1}^{N}\rho_n(D-2,m+1)\Big(\Pi^n(\mathcal{J})^{[n]} \;+\; \Pi^{n-1}\,\gamma\,\gamma\cdot(\mathcal{J})^{[n-1]}\Big)\;, \tag{3.51}$$

which projects the external current $\mathcal{J}$ to its transverse and γ-traceless part. This is achieved in two steps: the current $\mathcal{J}$ is first projected to its transverse part $\mathcal{J}^T$ using the projector Π introduced for bosonic fields, and then the γ trace is eliminated from the resulting expression via eq. (3.47). Notice that, as in the bosonic case, the presence of Π brings about the replacement of D with $D-2$ in this expression.

Simplifications are again possible in special circumstances, and in particular if $\mathcal{J}$ is transverse: eq. (3.51) reduces to

$$\mathcal{B}_c \mathcal{J} \ = \ \mathcal{J} \ + \ \sum_{n=1}^{N} \rho_n(D-2, m+1)(\eta^n \mathcal{J}^{[n]} \ + \ \eta^{n-1} \gamma \, \gamma \cdot \mathcal{J}^{[n-1]}) \ , \tag{3.52}$$

without the need for any explicit Π projectors, and hence with no explicit dependence on $\bar{p}$. Notice that $\mathcal{B}_c \mathcal{J}$ can also be written in the form

$$\mathcal{B}_c \mathcal{J} \ = \ \mathcal{T}_m^{(2)} \mathcal{J} \ + \ \rho_1(D, m) \ \gamma \ \gamma \cdot \mathcal{T}_m^{(3)} \mathcal{J} \ , \tag{3.53}$$

which makes it manifest that the current thus projected is *triply γ-traceless.* This is the counterpart of the condition (3.21) obtained for bosonic fields. As we shall see shortly, this presentation of the result is particularly useful when comparing with the propagator for the local unconstrained formulation of the previous Section.

In the Fang-Fronsdal theory with an external source $\mathcal{J}$ the field equation is

$$\mathcal{S} \ - \ \frac{1}{2} \, \eta \, \mathcal{S}' \ - \ \frac{1}{2} \, \gamma \, \not{\mathcal{S}} \ = \ \mathcal{J}, \tag{3.54}$$

which via the Bianchi identity implies the condition

$$\partial \cdot \mathcal{J} \ = \ - \ \frac{1}{2} \, \eta \, \partial \cdot \mathcal{S}' \ - \ \frac{1}{2} \, \gamma \, \partial \cdot \not{\mathcal{S}} \ . \tag{3.55}$$

As a result only the divergence of the γ-traceless part of $\mathcal{J}$ vanishes in general,

$$\mathcal{T}_{m-1}(\partial \cdot \mathcal{J}) = 0 \ . \tag{3.56}$$

In addition, the Fang-Fronsdal constraint on the triple γ-trace of ψ implies that $\mathcal{S}$, and hence $\mathcal{J}$ on account of eq. (3.54), are also triply γ-traceless. In analogy with what was done for bosons, eq. (3.54) can thus be inverted and $\mathcal{B} \mathcal{J}$ can be replaced with

$$\mathcal{B}_c \mathcal{J} \ = \ \mathcal{J} \ + \ \rho_1(D-2, m+1) \left(\eta \, \mathcal{J}' \ + \ \gamma \, \gamma \cdot \mathcal{J} \right) \ , \tag{3.57}$$

which in four space-time dimensions agrees with the expression given in [1].

In the unconstrained local formulation of the previous Section the situation is similar to some extent, since once the third of eqs. (2.52) is enforced, the first, the field equation for ψ, takes the form

$$\mathcal{W} \ - \ \frac{1}{2} \, \gamma \, \not{\mathcal{W}} \ - \ \frac{1}{2} \, \eta \, \mathcal{W}' \ = \ \mathcal{J} \ - \ \frac{1}{4} \, \eta \, \gamma \, \mathcal{Z} \ \equiv \ \mathcal{K} \ . \tag{3.58}$$

Just as for bosons the gauge invariant tensor $\mathcal{A}$ was doubly traceless on shell, so one can show that the gauge invariant spinor-tensor $\mathcal{W}$ is triply γ-traceless on shell,

$$\not{\mathcal{W}}' \ = \ 0 \ . \tag{3.59}$$

Hence, the same property holds for $\mathcal{K}$, so that the combination of $\mathcal{Z}$ and $\mathcal{J}$ results in

$$\mathcal{K} \ = \ \mathcal{T}_m^{(3)} \mathcal{J} \ , \tag{3.60}$$

that is therefore an effective Fang-Fronsdal current. One can then solve eq. (3.58), obtaining

$$\mathcal{W} \ = \ \mathcal{K} \ + \ \rho_1(D-2, m+1) \ [\gamma \, \not{\mathcal{K}} \ + \ \eta \, \mathcal{K}'] \ , \tag{3.61}$$

and therefore the current-current amplitude is determined by

$$\mathcal{K} \; + \; \rho_1(D-2, m+1) \, [\gamma \, \not{\mathcal{K}} \; + \; \eta \, \mathcal{K}'] \; . \tag{3.62}$$

In terms of $\mathcal{J}$, using (3.49), one can now conclude that

$$\mathcal{W} \; = \; \mathcal{T}_m^{(2)} \mathcal{J} \; + \; \rho_1(D,m) \, \gamma \, \gamma \cdot \mathcal{T}_m^{(3)} \mathcal{J} \; . \tag{3.63}$$

This expression is the direct counterpart of eq. (3.57), since, as we have stressed, $\rho_1(D,m) = \rho_1(D-2, m+1)$. It finally shows that in the unconstrained formulation the current exchange has the form (3.53), which indeed guarantees that the correct number of degrees of freedom propagates on-shell.

3.3 Current exchanges in an AdS background

In the last two Subsections we have shown that the constrained and unconstrained formulations for Bose or Fermi fields in flat space time are equivalent even in the presence of external currents. This result reflects the occurrence, in both settings, of tensors ($\mathcal{F}$ and $\mathcal{A}$, or $\mathcal{S}$ and $\mathcal{W}$) that on-shell are effectively subject to the (Fang-)Fronsdal (γ-)trace constraints and satisfy the same Bianchi identities. We have also identified effective currents (K or $\mathcal{K}$) that behave exactly like the Fang-Fronsdal currents. We now want to show briefly how this structure carries over to AdS backgrounds.

Let us begin by deriving the current exchange for unconstrained Bose fields in AdS. The key observation is that, when β is on shell, the first of eqs. (2.36) takes the form

$$\mathcal{A}_L \; - \; \frac{1}{2} \, g \, \mathcal{A}_L' \; + \; g^{\,2} \, \mathcal{C}_L \; = \; J \, , \tag{3.64}$$

where g denotes the AdS metric and, again, $\mathcal{A}_L'' = 0$. The same steps followed in the preceding Subsections for the flat-space analysis then lead to

$$\mathcal{A}_L \; = \; \sum_{n=0}^{N} \rho_n \, (D-2, \, s) \, g^{\,n} \, J^{\,[n]} \; . \tag{3.65}$$

In order to proceed further, it is very convenient to introduce the Lichnerowicz operator [32], that in an AdS background takes the form

$$\Box_L \, \varphi \; = \; \Box \, \varphi \; + \frac{1}{L^2} \, \left[s \, (D+s-2) \, \varphi \; - \; 2 \, g \, \varphi' \right] \; , \tag{3.66}$$

and is particularly convenient, since it commutes with contraction and covariant differentiation and allows one to write the AdS Fronsdal operator as

$$\left[\Box_L \; - \; \frac{2}{L^2} \, (s-1) \, (D+s-3) \right] \varphi \; - \; \nabla \, \nabla \cdot \varphi \; + \; \nabla^{\,2} \, \varphi' \; . \tag{3.67}$$

When contracted with a conserved current, clearly only the first term is relevant. As a result, the current exchange is finally determined by

$$\sum_{n=0}^{N} \rho_n \, (D-2, \, s) \, \frac{s!}{n! \, (s-2n)! \, 2^{\,n}} \; J^{\,[n]} \cdot \left[\Box_L \; - \; \frac{2}{L^2} (s-1)(D+s-3) \right]^{-1} J^{\,[n]} \; , \tag{3.68}$$

which reduces to the flat space amplitude in the limit $L \to \infty$.

Fermi fields in AdS, as we have seen, entail a few additional complications, since the current is now subject to a modified conservation law,

$$\nabla \cdot \mathcal{J} \,+\, \frac{1}{2L}\, \gamma \cdot \mathcal{J} \;=\; 0 \,. \tag{3.69}$$

When the auxiliary fields are on shell, the relevant field equation reads

$$\mathcal{W}_L \;-\; \frac{1}{2}\, \gamma \, \not{\mathcal{W}} \;-\; \frac{1}{2}\, \eta \, \mathcal{W}_L' \;=\; \mathcal{J} \;-\; \frac{1}{4}\, \eta \, \gamma \, \mathcal{Z}_L \;\equiv\; \mathcal{K}_L, \tag{3.70}$$

with, again, the triple γ-trace constraint $\not{\mathcal{W}}_L' \;=\; 0$. As in the flat case, one can first solve for $\mathcal{Z}_L$ and then obtain $\mathcal{W}_L$ from

$$\mathcal{W}_L \;=\; \mathcal{K}_L \;-\; \frac{1}{D+2(m-2)}\; \left[\gamma \not{\mathcal{K}}_L \;+\; \eta \, \mathcal{K}_L' \right] \;, \tag{3.71}$$

where the effective current $\mathcal{K}_L$ is subject to the triple γ-trace constraint $\mathcal{K}_L = \mathcal{T}_m^{(3)} \mathcal{J}$, with $\mathcal{T}_m$ defined as in (3.49) but for the replacement of the flat metric and the Dirac matrices with their AdS counterparts. Notice that $\mathcal{K}_L$ satisfies the modified conservation law

$$\mathcal{T}_m \left(\nabla \cdot \mathcal{K}_L \,+\, \frac{1}{2L}\, \not{\mathcal{K}}_L \right) \;=\; 0 \,. \tag{3.72}$$

One thus faces, again, an effective current $\mathcal{K}_L$ which is built from the conserved current $\mathcal{J}$ but nonetheless behaves as a constrained current of the Fang-Fronsdal theory. Constrained and unconstrained local formulations agree, the latter being effectively a Fang-Fronsdal theory with a partially conserved current which is built, as stressed above, from the conserved current $\mathcal{J}$.

4 Current exchanges in the non-local formulation

We can now turn to the non-local formulation. Confining our attention to bosonic fields, we begin by analyzing the current exchange in the non-local theory with reference to the Lagrangian equations proposed in [7], in order to display the problem. As anticipated, and as we shall see shortly, the Lagrangian equation proposed in [7] *is not* the proper counterpart of the local one discussed in Section 2 since, with standard couplings of the $\varphi \cdot J$ type, it does not result in the correct counting of degrees of freedom in current exchanges. We shall then present a unique set of non-local Lagrangian equations that, for all s, reproduce the current exchange of the previous Section. The result of this analysis will thus be a precise map, for all s, between the local and non-local formulations of the unconstrained theory. In the absence of external currents, however, these novel Lagrangian equations can also be turned into the non-Lagrangian equations of [7], eqs. (1.2) and (1.3). Here we confine our attention, for brevity, to the case of bosonic fields, but similar results are expected to hold for fermionic fields.

4.1 The problem

The iterative procedure of [7], reviewed in the previous Section, was meant to terminate, for a given value s of the spin, $s = 2n-1$ or $s = 2n$, after first reaching a non-local gauge invariant extension $\mathcal{F}^{(n)}$ of the Fronsdal operator. From this, as shown in [7], one can build a divergence-free Einstein-like tensor by simply combining $\mathcal{F}^{(n)}$ with its traces according to

$$\mathcal{G}^{(n)} \; = \; \sum_{p=0}^{n} \frac{(-1)^p \, (n-p)\,!}{2^p \; n\,!} \; \eta^p \; \mathcal{F}^{(n)\,[p]} \; . \tag{4.1}$$

The form of this Einstein-like tensor is determined by the Bianchi identity satisfied by the $\mathcal{F}^{(n)}$,

$$\partial \cdot \mathcal{F}^{(n)} \; - \; \frac{1}{2n} \; \partial \mathcal{F}^{(n)\;\prime} \; = \; - \; \left(1 + \frac{1}{2n}\right) \; \frac{\partial^{\,2n+1}}{\Box^{\;n-1}} \; \varphi^{\,[n+1]} \; , \tag{4.2}$$

where for the two relevant values of s associated to a given n, the term on the right-hand side vanishes identically. In this case eq. (4.2) has a number of interesting consequences, that can be derived taking successive traces:

$$\partial \cdot \mathcal{F}^{(n)\,[k]} \; - \; \frac{1}{2(n-k)} \; \partial \mathcal{F}^{(n)\,[k+1]} \; = \; 0 \; , \qquad (k \leq n-1) \; . \tag{4.3}$$

In particular, if the spin s is odd, so that $s = 2n-1$, one can see that

$$\partial \cdot \mathcal{F}^{(n)\,[n-1]} \; = \; 0 \; . \tag{4.4}$$

If the system is coupled to an external current $\mathcal{J}$, the Lagrangian field equations proposed in [7] thus read

$$\mathcal{G}^{\,(n)} \; \equiv \sum_{p=0}^{n} (-1)^p \, \frac{(n-p)!}{2^p \; n!} \; \eta^p \; \mathcal{F}^{\,(n)\,[p]} \; = \; \mathcal{J} \; , \tag{4.5}$$

where, as we have anticipated, $s = 2n-1$ or $s = 2n$. These equations can now be inverted noticing that

$$\rho_1(D-2n,s)\,\eta\,\mathcal{J}^{\,\prime} \; = \; - \sum_{p=1}^{n} (-1)^p \; \frac{p\;(n-p)!}{2^p n!} \; \eta^p \, \mathcal{F}^{\,(n)\,[p]} \; , \tag{4.6}$$

$$\rho_2(D-2n,s)\,\eta^2 \mathcal{J}^{\,\prime\prime} \; = \; \sum_{p=2}^{n} (-1)^p \, \frac{p(p-1)}{2} \; \frac{(n-p)!}{2^p n!} \; \eta^{\,p} \; \mathcal{F}^{(n)[p]} \; , \tag{4.7}$$

and continuing in this fashion one readily obtains

$$\mathcal{F}^{(n)} \; = \; \sum_{n=0}^{n} \; \rho_n(D-2n,s) \; \eta^n \; \mathcal{J}^{[n]}, \tag{4.8}$$

where the coefficients ρ_n were introduced in Section 3.

One should now contract eq. (4.8) with a conserved current, and the occurrence of a key simplification may be simply anticipated: the relations $\mathcal{J} \cdot \mathcal{F}^{(k)} = \ldots = \mathcal{J} \cdot \mathcal{F}$ hold, up to terms involving the vanishing divergence of $\mathcal{J}$. The current exchange in the non-local theory of [7], based on the Einstein-like tensor of eq. (4.5), is thus determined by

$$\sum_{n=0}^{n} \rho_n(D-2n,s)\,\eta^n \; \mathcal{J}^{[n]} \; , \tag{4.9}$$

and clearly *disagrees* with eq. (3.18) whenever $s > 2$, *i.e.* for all interesting cases of higher-spin fields.

In order to better appreciate the nature of the problem, it is instructive to take a closer look at the relatively simple but still non-trivial case of a spin-3 field. The kinetic operator of [7],

$$\mathcal{F}^{(2)}_{\mu\nu\rho} \;=\; \mathcal{F}_{\mu\nu\rho} \,-\, \frac{1}{3\,\Box}\left(\partial_\mu\,\partial_\nu\,\mathcal{F}'_\rho \,+\, \partial_\nu\,\partial_\rho\,\mathcal{F}'_\mu \,+\, \partial_\rho\,\partial_\mu\,\mathcal{F}'_\nu\right)\,, \tag{4.10}$$

is then determined by a single iteration of eq. (3.1), and satisfies the Bianchi identity

$$\partial\cdot\mathcal{F}^{(2)} \;-\; \frac{1}{4}\,\partial\,\mathcal{F}^{(2)\,\prime} \,=\, 0\,, \tag{4.11}$$

to be compared with the conventional Bianchi identity for the Fronsdal operator

$$\partial\cdot\mathcal{F} \,-\, \frac{1}{2}\,\partial\,\mathcal{F}' \,=\, 0\,. \tag{4.12}$$

In both cases the Lagrangian field equations would couple the corresponding divergence-free Einstein-like tensors to divergence-free currents, according to

$$\mathcal{G} \;\equiv\; \mathcal{F} \;-\; \frac{1}{2}\,\eta\,\mathcal{F}' \;=\; J\,, \tag{4.13}$$

$$\mathcal{G}^{(2)} \;\equiv\; \mathcal{F}^{(2)} \;-\; \frac{1}{4}\,\eta\,\mathcal{F}^{(2)\,\prime} \;=\; \mathcal{J}\,, \tag{4.14}$$

but the definitions of $\mathcal{G}$ and $\mathcal{G}^{(1)}$ involve different coefficients, reflecting the differences between the Bianchi identities of eqs. (4.11) and (4.12). Inverting these equations, one would then arrive, in the two cases, at the current exchanges

$$J\cdot J \;-\; \frac{3}{D}\,J'\cdot J'\,, \tag{4.15}$$

$$\mathcal{J}\cdot\mathcal{J} \;-\; \frac{3}{D-2}\,\mathcal{J}'\cdot\mathcal{J}'\,, \tag{4.16}$$

which are clearly different, as special cases of eqs. (3.18) and (4.9). The lesson to be drawn from this example is that the differences between the current exchanges in the local formulation of the previous Sections and in the non-local formulation of [7] based on eq. (4.5) reflect those between the modified Bianchi identity satisfied by the non-local kinetic operators $\mathcal{F}^{(n)}$ and the original Bianchi identity satisfied by the Fronsdal operator $\mathcal{F}$.

Interestingly, the two results obtained in this spin-3 example can actually be mapped into one another provided the currents J and $\mathcal{J}$ are related by a suitable *non-local* field redefinition. Indeed, defining

$$\bar{\eta} \;=\; \eta \;-\; \frac{\partial^2}{\Box}\,, \tag{4.17}$$

a non-local extension of η which is *divergence free*, if $\mathcal{J}$ and J are related according to

$$\mathcal{J} \;=\; J \;+\; \frac{-3D\pm\sqrt{3D^2-6D}}{3D(D+1)}\;\bar{\eta}\;J'\,, \tag{4.18}$$

eq. (4.16) turns precisely into (4.15). Notice that the map between the two constructions thus obtained is compatible with the conservation of both J and $\mathcal{J}$.

This result is adding a useful piece of information: while the tensor $\mathcal{G}^{(1)}$ does not couple as expected to an external current, a suitable non-local combination of this tensor with its trace does. These more singular objects satisfy Bianchi identities that are closer to that satisfied by

$\mathcal{F}$ and $\mathcal{A}$: hence, they are precisely the types of non-local kinetic operators we are after, for all values of s. As a side technical remark, let us stress again that, strictly speaking, neither $\bar{\eta}$ nor the $\mathcal{F}^{(n)}$ are well-defined quantities on shell. Hence, it should be understood that, in all the present treatment of current exchanges in the non-local formulation, one is actually working off shell all the way and only the final results are continued on-shell to arrive at the correct counting of degrees of freedom.

Actually, in [7] a more singular class of non-local operators was also considered. For all s, they can be obtained combining the $\mathcal{F}^{(n)}$ with their traces and are quite interesting, since they lead in general to field equations of the type

$$\tilde{\mathcal{F}} \equiv \mathcal{F} \,-\, 3\,\partial^{\,3}\,\mathcal{H} \;=\; 0 \;, \tag{4.19}$$

that have the same form as the non-Lagrangian compensator equations (2.19). In particular, for $s = 3$ one can define

$$\tilde{\mathcal{F}}^{(1)}_{\mu\nu\rho} \;\equiv\; \mathcal{F}_{\mu\nu\rho} \;-\; \frac{\partial_\mu\partial_\nu\partial_\rho}{\Box^2}\;\partial\cdot\mathcal{F}^{\,\prime} \;, \tag{4.20}$$

which actually satisfies the same Bianchi identity as $\mathcal{F}$, so that the corresponding Einstein tensor is

$$\tilde{\mathcal{G}} \;=\; \tilde{\mathcal{F}} \;-\; \frac{1}{2}\,\eta\,\tilde{\mathcal{F}}^{\,\prime} \;. \tag{4.21}$$

Since $\mathcal{F}$ and $\tilde{\mathcal{F}}$ differ by terms that vanish upon contraction with a conserved current, with this type of non-local Lagrangian equation the current exchange clearly agrees with the result obtained in the previous Subsection for the local theory. Notice that in this case one could also arrive at eq. (4.20) starting from the two conditions

$$\begin{aligned} &\mathcal{A} \;-\; \frac{1}{2}\,\eta\,\mathcal{A}^{\,\prime} \;=\; 0 \\ &\partial\cdot\mathcal{A}^{\,\prime} \;=\; 0 \;, \end{aligned} \tag{4.22}$$

where the second guarantees that the first couple consistently to a conserved current.

The lesson to be drawn from this example is, therefore, that for a given s, equal to $2n-1$ o $2n$, the Einstein-like tensors that correspond to those of the local theory *are not* directly expressible in terms of the single $\mathcal{F}^{(n)}$ operator, but are to involve other more singular operators $\mathcal{F}^{(m)}$, with $m > n$. The issue is now to determine the form of these Einstein-like tensors for all s and to elucidate the corresponding geometrical structure. As in other circumstances, for instance for the Lagrangians of Section 2, one has to reach the $s = 4$ case to first uncover the relevant pattern. In this case, letting

$$\mathcal{A} \,=\, \mathcal{F} \,-\, 3\,\partial^{\,3}\,\alpha_{\,\varphi}\,,$$

and insisting on the condition $\partial\cdot\mathcal{A}^{\,\prime} = 0$, leads to

$$\alpha_{\,\varphi} \;=\; \frac{1}{3\,\Box^{\,2}}\,\partial\cdot\mathcal{F}^{\prime} \;-\; \frac{\partial}{4\,\Box^{\,3}}\,\partial\cdot\partial\cdot\mathcal{F}^{\prime}\,. \tag{4.23}$$

On the other hand, combining it with its trace, one can turn the non-local equation $\mathcal{F}^{(1)} = 0$ into

$$\mathcal{T} \,\equiv\, \mathcal{F} \,-\, 3\,\partial^{\,3}\,\mathcal{H}_{\,\varphi} \,=\, 0\,, \tag{4.24}$$

where

$$\mathcal{H}_{\,\varphi} \;=\; \frac{1}{3\,\Box^{2}}\,\partial\cdot\mathcal{F}^{\prime} \;-\; \frac{\partial}{4\,\Box^{\,2}}\,\mathcal{F}^{\prime\prime}\,, \tag{4.25}$$

Notice that α_φ and $\mathcal{H}_\varphi$ *do not coincide.* Their difference, however, is determined by a gauge invariant quantity according to

$$\alpha_\varphi - \mathcal{H}_\varphi = \frac{3}{4}\frac{\partial}{\Box}(\varphi'' - 4\partial\cdot\alpha_\varphi) , \tag{4.26}$$

a relation that can also be presented in the more symmetric form

$$\varphi'' - 4\partial\cdot\alpha_\varphi = \frac{1}{4}\left[\varphi'' - 4\partial\cdot\mathcal{H}_\varphi\right] , \tag{4.27}$$

and implies that the two choices result in the two inequivalent Einstein tensors

$$\begin{aligned} E_\alpha &= \mathcal{A} - \frac{1}{2}\eta\left(\mathcal{A}' - \frac{1}{3}\frac{\partial^2}{\Box}\mathcal{A}''\right) - \frac{1}{6}\eta^2\mathcal{A}'' , \\ E_\mathcal{H} &= \mathcal{T} - \frac{1}{2}\eta\left(\mathcal{T}' - \frac{1}{3}\frac{\partial^2}{\Box}\mathcal{T}''\right) + \frac{5}{24}\eta^2\mathcal{T}'' . \end{aligned} \tag{4.28}$$

Notice that none of the two choices results in the correct current exchange. The first indeed gives

$$\mathcal{J}\cdot\mathcal{A} = \mathcal{J}\cdot\mathcal{J} - \frac{6}{D+2}\mathcal{J}'\cdot\mathcal{J}' + \frac{3(D+6)}{(D+2)(D^2+6D+2)}\left(\mathcal{J}''\right)^2 , \tag{4.29}$$

while the second gives

$$\mathcal{J}\cdot\mathcal{T} = \mathcal{J}\cdot\mathcal{J} - \frac{6}{D+2}\mathcal{J}'\cdot\mathcal{J}' + \frac{3(6-5D)}{(D+2)(-5D^2+6D+8)}\left(\mathcal{J}''\right)^2 , \tag{4.30}$$

to be compared with the correct result, given in eq. (3.19).

4.2 Nonlocal equations with a proper current exchange

The arguments of the previous Subsection show clearly that the Einstein tensor is the crucial ingredient behind the current amplitude, whose form is determined by the Bianchi identity for $\mathcal{A}_{nl}$, the gauge-invariant extension of the Fronsdal operator $\mathcal{F}$. Actually, a closer look at the discussion of Section 3 for the local theory shows that the non-local formulation can reproduce the correct current-current amplitude provided its Lagrangian field equations

$$\mathcal{G}_{nl} = \mathcal{J} \tag{4.31}$$

lead to the solution

$$\mathcal{A}_{nl} = \mathcal{P}_c\mathcal{J} , \tag{4.32}$$

where $\mathcal{P}_c$ was defined in eq. (3.20). Eqs. (3.21), however, imply a pair of consistency conditions for this statement,

$$\begin{aligned} \mathcal{A}_{nl}'' &= 0 , \\ \mathcal{A}_{nl} - \frac{1}{2}\eta\mathcal{A}_{nl}' &= \mathcal{J} + \eta^2(\ldots) , \end{aligned} \tag{4.33}$$

which, as we have seen, are precisely met by the local construction.

One is thus led to search for an Einstein tensor that differs from that of eq. (4.5), and is rather of the form

$$\mathcal{G}_{nl} = \mathcal{A}_{nl} - \frac{1}{2}\eta\mathcal{A}_{nl}' + \eta^2\mathcal{B} , \tag{4.34}$$

for some tensor $\mathcal{B}$, and where $\mathcal{A}''_{nl} = 0$. Conversely, given this form of the Einstein tensor, with a doubly traceless $\mathcal{A}_{nl}$, one can see that the solution of eq. (4.31) is bound to take the form (4.32). On the other hand, the Einstein tensor can take the form of eq. (4.34) only if $\mathcal{A}_{nl}$ obeys the Bianchi identity

$$\partial \cdot \mathcal{A}_{nl} \; - \; \frac{1}{2} \, \partial \mathcal{A}'_{nl} \; = \; 0 \; , \tag{4.35}$$

and if the divergence of the trace of $\mathcal{A}_{nl}$ is a pure gradient, a condition that we shall write in the form

$$\partial \cdot \mathcal{A}'_{nl} \; = \; 2 \, \partial \, \mathcal{D}_{nl} \; , \tag{4.36}$$

with $\mathcal{D}_{nl}$ an arbitrary non-local tensor. These two requirements are in fact necessary to guarantee the consistency of eq. (4.31) with the conservation of the current $\mathcal{J}$.

Eqs. (4.35) and (4.36) are also sufficient if they are combined with the additional demand that $\mathcal{A}_{nl}$ be of the form

$$\mathcal{A}_{nl} \; = \; \mathcal{F} \; - \; 3 \, \partial^{\,3} \, \alpha_{nl}, \tag{4.37}$$

for some α_{nl} to be determined. In order to prove that this is actually the case, notice first that the Bianchi identity reads

$$\partial \cdot \mathcal{A}_{nl} \; - \; \frac{1}{2} \, \partial \, \mathcal{A}'_{nl} \; = \; - \, \frac{3}{2} \, \partial^{3} \, \left(\varphi \, '' \; - \; 4 \partial \cdot \alpha_{nl} \; - \; \partial \, \alpha'_{nl} \right) \; , \tag{4.38}$$

so that it vanishes provided α_{nl} solves the constraint

$$\mathcal{C}_{nl} \; \equiv \; \left(\varphi \, '' \; - \; 4 \partial \cdot \alpha_{nl} \; - \; \partial \, \alpha'_{nl} \right) \; = \; 0 \; . \tag{4.39}$$

Since the double trace of $\mathcal{A}_{nl}$ can be expressed as

$$\mathcal{A}''_{nl} \; = \; 3 \, \Box \, \mathcal{C}_{nl} \; + \; 3 \, \partial \, \partial \cdot \mathcal{C}_{nl} \; + \; \partial^{\,2} \, \mathcal{C}'_{nl} \; , \tag{4.40}$$

the condition (4.39) also guarantees that A_{nl} be doubly traceless on shell. However, eq. (4.36) and its successive traces also imply that

$$\mathcal{G}_{nl} \; = \; \mathcal{A}_{nl} \; - \; \frac{1}{2} \, \eta \, \mathcal{A}'_{nl} \; + \; \eta^{\,2} \, \mathcal{D}_{nl} \; + \; \ldots + \eta^{\,n+2} \, \frac{\mathcal{D}^{[n]}_{nl}}{2^{n} \, n!}, \tag{4.41}$$

where n is the integer such that $s = 2(n+2)$ or $s = 2n+5$, is indeed conserved and is of the form (4.34).

To summarize, the current-current amplitude is correctly reproduced by the modification (4.37) of the Fronsdal operator provided the two conditions (4.39) and (4.36) are met. Notice that (4.39) and its consequence (4.40) are the counterparts of the β equation of motion in the local formulation (the third equation in (2.17)). On the other hand, eq. (4.36) can be regarded as the counterpart of the α equation of motion of the local unconstrained formulation, while the very form of the Einstein tensor (4.34) is clearly tailored after the one entering the φ equations of motion in the local formulation.

Now we would like to show that a solution to these two conditions exists and is unique. To this end, we shall first determine α_{nl} in terms of φ. We shall then conclude the present Subsection by displaying the geometrical structures underlying $\mathcal{A}_{nl}$.

Let us first notice that

$$
\begin{aligned}
\partial \cdot \mathcal{A}'_{nl} =& \, 3 \, \Box \, \partial \cdot \varphi' \, - \, 2 \, \partial \cdot \partial \cdot \partial \cdot \varphi \, - \, 3 \, \Box^{\,2} \, \alpha_{nl} \\
& - \, \partial \left(9 \, \Box \, \partial \cdot \alpha_{nl} \, + \, 3 \, \partial \, \partial \cdot \partial \cdot \alpha_{nl} \right. \\
& \left. + \, \frac{3}{2} \, \Box \, \partial \, \alpha'_{nl} \, + \, \partial^{\,2} \, \partial \cdot \alpha'_{nl} \, - \, \partial \cdot \partial \cdot \varphi' \, - \, \Box \, \varphi'' \, - \, \frac{1}{2} \, \partial \cdot \varphi'' \right) \, .
\end{aligned}
\tag{4.42}
$$

Requiring that this expression be equal to $2 \, \partial \, \mathcal{D}_{nl}$ gives the two conditions

$$
\begin{aligned}
& 3 \, \Box \, \partial \cdot \varphi' \, - \, 2 \partial \cdot \partial \cdot \partial \cdot \varphi \, - \, 3 \, \Box^{\,2} \, \alpha_{nl} \, = \, \partial \, f \\
& \partial \, \left(9 \, \Box \, \partial \cdot \, \alpha_{nl} \, + \, 3 \partial \, \partial \cdot \partial \cdot \alpha_{nl} \, + \, \frac{3}{2} \, \Box \, \partial \, \alpha'_{nl} \right. \\
& \left. + \, \partial^{\,2} \, \partial \cdot \alpha'_{nl} \, - \, \partial \cdot \partial \cdot \varphi' \, - \, \Box \, \varphi'' \, - \, \frac{1}{2} \, \partial \, \partial \cdot \varphi'' \right) \, = \, f \, - \, 2 \, \mathcal{D}_{nl} \; ,
\end{aligned}
\tag{4.43}
$$

where f is an arbitrary tensor, that will be determined shortly by eq. (4.39). The first equation gives indeed

$$
\alpha_{nl} \, = \, \frac{1}{\Box} \, \partial \cdot \varphi' \, - \, \frac{2}{3 \, \Box^{\,2}} \, \partial \cdot \partial \cdot \partial \cdot \varphi \, - \, \frac{\partial}{3 \, \Box^{\,2}} f \; ,
\tag{4.44}
$$

and inserting this expression in eq. (4.39) yields an equation for f,

$$
\begin{aligned}
f = & - \frac{3}{4} \, \Box \, \varphi'' \, - \, \frac{2}{\Box} \, \partial \cdot \partial \cdot \partial \cdot \partial \cdot \varphi \, + \, 3 \, \partial \cdot \partial \cdot \varphi' \\
& + \, \frac{3}{4} \, \partial \, \partial \cdot \varphi'' \, - \, \frac{1}{2 \Box} \, \partial \, \partial \cdot \partial \cdot \partial \cdot \varphi' \, - \, \frac{3}{2} \, \frac{\partial}{\Box} \, \partial \cdot f \, - \, \frac{1}{2} \, \frac{\partial^{\,2}}{\Box} \, f' \; .
\end{aligned}
\tag{4.45}
$$

One can now look for a solution of the form

$$
f = \sum \partial^{\,n} \, f_n \; ,
\tag{4.46}
$$

where for a spin-s field the sum terminates at $n = s - 4$, to be determined by successive iterations, and the result is

$$
f_0 \, = \, - \, \frac{3}{4} \, \Box \, \varphi'' \, - \, \frac{2}{\Box} \, \partial \cdot \partial \cdot \partial \cdot \partial \cdot \varphi \, + \, 3 \, \partial \cdot \partial \cdot \varphi'
\tag{4.47}
$$

$$
f_1 \, = \, \frac{3}{4} \, \partial \cdot \varphi'' \, - \, \frac{2}{\Box} \, \partial \cdot \partial \cdot \partial \cdot \varphi' \, + \, \frac{6}{5 \, \Box^2} \, \partial \cdot \partial \cdot \partial \cdot \partial \cdot \partial \cdot \varphi
\tag{4.48}
$$

$$
f_n \, = \, - \, \frac{1}{(n+1) \, (n+4)} \, \frac{1}{\Box} \, \Big(2 \, n \, (n+2) \, \partial \cdot f_{n-1} \, + \, n \, (n-1) \, f'_{n-2} \Big) \quad (n \geq 2) \; .
\tag{4.49}
$$

This truncation has an interesting consequence: if α_s denotes the expression for α_{nl} for a spin s field, taking into account the last terms leads to the recursion relation

$$
\alpha_{s+1} \, = \, \alpha_s \, - \, \frac{s-2}{3} \, \frac{\partial^{\,s-2}}{\Box^2} \, f_{s-3} \; .
\tag{4.50}
$$

The solutions for the first few cases are

$$
\alpha_3 \, = \, \frac{1}{\Box} \, \partial \cdot \varphi' \, - \, \frac{2}{3 \, \Box^{\,2}} \, \partial \cdot \partial \cdot \partial \cdot \varphi
\tag{4.51}
$$

$$
\alpha_4 \, = \, \alpha_3 \, + \, \frac{\partial}{\Box^{\,2}} \, \Big(\frac{1}{4} \, \Box \, \varphi'' \, + \, \frac{2}{3 \, \Box} \, \partial \cdot \partial \cdot \partial \cdot \partial \cdot \varphi \, + \, \partial \cdot \partial \cdot \varphi' \Big)
\tag{4.52}
$$

$$
\alpha_5 \, = \, \alpha_4 \, - \, \frac{\partial^{\,2}}{\Box^{\,2}} \, \Big(\frac{1}{2} \, \partial \cdot \varphi'' \, - \, \frac{4}{3 \, \Box} \, \partial \cdot \partial \cdot \partial \cdot \varphi' \, + \, \frac{4}{5 \, \Box^{\,2}} \, \partial \cdot \partial \cdot \partial \cdot \partial \cdot \partial \cdot \varphi \Big) \; ,
\tag{4.53}
$$

and can also be expressed in terms of the Fronsdal operator as

$$\alpha_3 = \frac{1}{3\,\Box^2}\,\partial\cdot\mathcal{F}'\,, \tag{4.54}$$

$$\alpha_4 = \frac{1}{3\,\Box^2}\,\partial\cdot\mathcal{F}' - \frac{\partial}{3\,\Box^3}\,\partial\cdot\partial\cdot\mathcal{F}' + \frac{1}{12}\,\frac{\partial}{\Box^2}\,\mathcal{F}''\,, \tag{4.55}$$

$$\alpha_5 = \frac{1}{3\,\Box^2}\,\partial\cdot\mathcal{F}' - \frac{\partial}{3\,\Box^3}\partial\cdot\partial\cdot\mathcal{F}' + \frac{2}{5}\frac{\partial^2}{\Box^4}\partial\cdot\partial\cdot\partial\cdot\mathcal{F}' + \frac{1}{12}\frac{\partial}{\Box^2}\,\mathcal{F}'' - \frac{1}{5}\frac{\partial^2}{\Box^3}\,\partial\cdot\mathcal{F}''. \tag{4.56}$$

The Einstein tensor is finally determined by α_{nl} and $\mathcal{D}_{nl}$ given in (4.43). For spin $s = 4$ and $s = 5$, for instance, the results read

$$\mathcal{D}_4 = \frac{1}{2}\left(\frac{1}{\Box}\partial\cdot\partial\cdot\mathcal{F}' - \mathcal{F}''\right) \tag{4.57}$$

$$\begin{aligned}\mathcal{D}_5 &= \frac{1}{2}\left(\frac{1}{\Box}\,\partial\cdot\partial\cdot\mathcal{F}' - \mathcal{F}'' - \frac{\partial}{\Box^2}\,\partial\cdot\partial\cdot\partial\cdot\mathcal{F}' + \frac{\partial}{\Box}\,\partial\cdot\mathcal{F}''\right)\\ &= \frac{1}{2}\left(1 - \frac{\partial\,\partial\cdot}{\Box}\right)\left(\frac{1}{\Box}\,\partial\cdot\partial\cdot\mathcal{F}' - \mathcal{F}''\right).\end{aligned} \tag{4.58}$$

There is actually a more illuminating way to proceed. The key idea is to present the solution in a manifestly gauge invariant fashion by expressing $\mathcal{A}_{nl}$ in terms of the higher-spin curvatures. To this end, let us begin by considering the geometric operators

$$\mathcal{F}^{(n+1)} = \begin{cases}\frac{1}{\Box^n}\,\mathcal{R}^{[n+1]} & s = 2\,(n+1)\,,\\ \frac{1}{\Box^n}\,\partial\cdot\mathcal{R}^{[n]} & s = 2\,n+1\,.\end{cases} \tag{4.59}$$

As discussed in [7], these operators are directly related to the $\mathcal{F}^{(n)}$ of eq. (3.1), and hence satisfy the sequel of identities

$$\partial\cdot\mathcal{F}^{(n+1)\,[k]} = \frac{1}{2\,(n-k+1)}\,\partial\,\mathcal{F}^{(n+1)\,[k+1]}\,, \tag{4.60}$$

so that, in the odd case, $\partial\cdot\mathcal{F}^{(n+1)\,[n]} \equiv 0$.

As a result, all divergences of the tensors $\mathcal{F}_{n+1}$ can be expressed in terms of traces, so that the only available independent structures are

$$\mathcal{F}^{(n+1)},\quad \mathcal{F}^{(n+1)\,\prime},\quad \mathcal{F}^{(n+1)\,\prime\prime},\quad \ldots,\quad \mathcal{F}^{(n+1)\,[k]},\quad \ldots,\quad \mathcal{F}^{(n+1)\,[q]}\,, \tag{4.61}$$

where $q = n+1$ or $q = n$ depending on whether the rank is $s = 2\,(n+1)$ or $s = 2\,n+1$. One can now construct the most general linear combination of *all* these terms, with arbitrary coefficients (the first may be set to one, up to an overall normalization)

$$\mathcal{A}_{nl} = \mathcal{F}^{(n+1)} + a_1\,\frac{\partial^2}{\Box}\,\mathcal{F}^{(n+1)\,\prime} + \ldots + a_k\,\frac{\partial^{2k}}{\Box^k}\,\mathcal{F}^{(n+1)\,[k]} + \ldots + \begin{cases} a_{n+1}\,\frac{\partial^{2\,(n+1)}}{\Box^{n+1}}\,\mathcal{F}^{(n+1)\,[n+1]} & s = 2\,(n+1)\,,\\ a_n\,\frac{\partial^{2\,n}}{\Box^{n}}\,\mathcal{F}^{(n+1)\,[n]} & s = 2\,n+1\,.\end{cases} \tag{4.62}$$

One can first determine a_1 by requiring that $\mathcal{A}_{nl}$ be of the form (4.37). The remaining coefficients may then be fixed, recursively, demanding that the proper Bianchi identity,

$$\partial\cdot\mathcal{A}_{nl} - \frac{1}{2}\,\partial\,\mathcal{A}_{nl}' = 0\,, \tag{4.63}$$

hold for A_{nl} or, equivalently, that A_{nl} be doubly traceless. Indeed, demanding that all terms in $\partial^{\,2}$ disappear yields the condition

$$a_1 \;=\; \frac{n}{n\,+\,1}\ , \tag{4.64}$$

while the double trace $\mathcal{A}''$ vanishes provided

$$a_{k+2} \;=\; -\,\frac{n\,+\,k+1}{n\,-\,k}\,\left\{\frac{n\,+\,k}{n\,-\,k\,+\,1}\,a_k\,+2\,a_{k+1}\right\}\ , \tag{4.65}$$

so that there is one and only one solution to our conditions.

The explicit solution of this difference equation may be foreseen after a few iterations, and reads

$$a_k \;=\; (-1)^{k+1}\,(2\,k\,-\,1)\,\frac{n\,+\,2}{n\,-\,1}\,\prod_{j=-1}^{k-1}\,\frac{n\,+\,j}{n\,-\,j\,+\,1}\,, \tag{4.66}$$

so that

$$\mathcal{A}_{nl} \;=\; \sum_{k=0}^{n+1}(-1)^{k+1}\,(2\,k\,-\,1)\,\left\{\frac{n\,+\,2}{n\,-\,1}\,\prod_{j=0}^{k-1}\,\frac{n\,+\,j}{n\,-\,j\,+\,1}\right\}\,\frac{\partial^{\,2k}}{\Box^{\,k}}\,\mathcal{F}^{(n+1)\,[k]}\ . \tag{4.67}$$

Alternatively, the Bianchi identity (4.63) holds provided

$$a_{k+1} \;=\; -\,a_k\,\frac{(2k+1)(n+k)}{(2k-1)(n-k+1)}\ , \tag{4.68}$$

which is also solved by eq. (4.66).

It is remarkable that (4.68) holds *without any assumption on the form of the tensor* $\mathcal{A}_{nl}$, aside from the condition that no η's be present in the linear combination (4.62). This means that by simply imposing the Bianchi identity on $\mathcal{A}_{nl}$ both the double-tracelessness and the compensator form $\mathcal{A}_{nl}\;=\;\mathcal{F}\;-\;3\,\partial^{\,3}\alpha_{nl}$ emerge as direct consequences.

To these results one should finally add the explicit solution for $\mathcal{D}_{nl}$,

$$\mathcal{D}_{nl} \;=\; \frac{1}{2}\,\sum_{k=2}^{n+1}\,a_k\,\left\{\frac{1}{2\,k\,-\,3}\,\frac{\partial^{\,2\,(k-2)}}{\Box^{\,k-2}}\,\mathcal{F}^{(n+1)\,[k]}\,+\,\frac{2n+4k+1}{2(2k-1)\,(n-k+1)}\,\frac{\partial^{\,2\,(k-1)}}{\Box^{\,k-1}}\,\mathcal{F}^{(n+1)\,[k+1]}\right.$$
$$\left.+\,\frac{n+k+1}{2(n-k)\,(n-k+1)}\,\frac{\partial^{\,2\,k}}{\Box^{\,k}}\,\mathcal{F}^{(n+1)\,[k+2]}\right\} \tag{4.69}$$

which allows to complete the construction of non-local Einstein-like tensors leading to correct current exchanges. In order to show that these tensors are indeed divergence-free, in the odd-spin case, after all possible traces of $\mathcal{D}_{nl}$ are computed and all divergences are turned into further traces via the Bianchi identities, one can notice that the result only involves divergences of $\mathcal{F}^{(n+1)\,[n]}$, that vanish identically because of eq. (4.60).

As an example, the modified Fronsdal tensors for the cases of spin $s = 3, 4$ and 5 read

$$\begin{aligned}
\mathcal{A}_3 \;&=\; \frac{1}{\Box}\,\partial\cdot\mathcal{R}'\,+\,\frac{\partial^{\,2}}{2\,\Box^{\,2}}\,\partial\cdot\mathcal{R}''\,, && (4.70)\\
\mathcal{A}_4 \;&=\; \frac{1}{\Box}\,\mathcal{R}''\,+\,\frac{1}{2}\,\frac{\partial^{\,2}}{\Box^{\,2}}\,\mathcal{R}'''\,-\,3\,\frac{\partial^{\,4}}{\Box^{\,3}}\,\mathcal{R}^{\,[4]}\,, && (4.71)\\
\mathcal{A}_5 \;&=\; \frac{1}{\Box^{\,2}}\,\partial\cdot\mathcal{R}''\,+\,\frac{2}{3}\,\frac{\partial^{\,2}}{\Box^{\,3}}\,\partial\cdot\mathcal{R}'''\,-\,3\,\frac{\partial^{\,4}}{\Box^{\,4}}\,\partial\cdot\mathcal{R}^{\,[4]}\,, && (4.72)
\end{aligned}$$

while the corresponding $\mathcal{D}$ tensors are similarly given by

$$\mathcal{D}_4 \;=\; -\frac{3}{8}\frac{1}{\Box}\,\mathcal{R}^{[4]}\,, \tag{4.73}$$

$$\mathcal{D}_5 \;=\; -\,\frac{5}{8\,\Box^{\,2}}\,\partial\cdot\mathcal{R}^{[4]}\,. \tag{4.74}$$

In the absence of sources, taking successive traces of the Lagrangian equation of motion coming from (4.34)

$$\mathcal{A}_{nl} \,-\, \frac{1}{2}\,\eta\,\mathcal{A}_{nl}' \,+\, \eta^{\,2}\,\mathcal{B} \,=\, 0\,, \tag{4.75}$$

one can reduce it to $\mathcal{A}_{nl} \,=\, 0$. This last equation, in turn, can be shown to imply the non-lagrangian equations of [7], eqs. (1.2) and (1.3), by making careful use of the Bianchi identities (4.60). For example, for the spin 4 case, once $\mathcal{B}$ and $\mathcal{A}_{nl}'$ are shown to vanish one is left with

$$\mathcal{F}^{(2)} \,+\, \frac{1}{2}\,\frac{\partial^{\,2}}{\Box}\,\mathcal{F}^{(2)\,\prime} \,-\, 3\,\frac{\partial^{\,4}}{\Box^{\,2}}\,\mathcal{F}^{(2)\,\prime\prime} \,=\, 0\,, \tag{4.76}$$

whose trace implies

$$\mathcal{F}^{(2)\,\prime} \,-\, \frac{\partial^{\,2}}{\Box}\,\mathcal{F}^{(2)\,\prime\prime} \,=\, 0\,. \tag{4.77}$$

Taking the divergence of this last relation, and using the identity $\partial\cdot\mathcal{F}^{(2)\,\prime} \,=\, \frac{1}{2}\,\partial\mathcal{F}^{(2)\,\prime\prime}$ it is then possible to show that $\mathcal{F}^{(2)\,\prime\prime} \,=\, 0$, which implies, via eq. (4.77), that $\mathcal{F}^{(2)\,\prime} \,=\, 0$ and finally $\mathcal{F}^{(2)} \,=\, 0$, eq. (1.3), as previously advertised.

5 Conclusions

In this paper we have examined a number of problems that present themselves when higher-spin gauge fields interact with external currents, the key motivation for our analysis being the precise comparison between Fronsdal's constrained formulation of [1] and the unconstrained Lagrangian formulations of [7] and [18]. To this end, we have began by streamlining the results of [18], that are here presented in terms of a few invariant structures which also extend rather simply to the interesting cases of *AdS* backgrounds. The subsequent Sections have dealt with the precise comparison between the different available formulations in the presence of external sources. These provide an important testing ground for the Lagrangian equations, and indeed while the local formulation of [18] is directly equivalent to Fronsdal's constrained formulation of [1], this is not the case for the Lagrangian equation proposed in [7]. In Section 4 we have thus displayed the precise map between the minimal unconstrained local Lagrangian formulation for fully symmetric tensors and spinor-tensors presented in [18] and the non-local formulation of [7], and we have proposed a new set of non-local Lagrangian equations that lead to a proper current exchange and, in the free case, can be reduced to the non-Lagrangian equations (1.2) and (1.3). Let us stress that the non-local geometric form thus identified is uniquely determined by the procedures described in Section 4.

NOTE ADDED

The current exchange of eq. (3.46) and its fermionic counterpart determine the van Dam-Veltman-Zakharov discontinuity [33] in flat space for this whole classes of (bosonic and fermionic) higher-spin fields. These expressions depend on the dimension D of space time, and the discontinuity follows directly from the comparison of the D-dimensional result with the corresponding one in $D+1$ dimensions. This is the case since the massless theory in $D+1$ dimensions, after a suitable reduction à la Scherk-Schwarz [34], describes irreducibly a massive field in the Stueckelberg mode. The extension of this result to an AdS background, along the lines of what was done by Higuchi and Porrati for $s = 2$ [35], is quite interesting and will be discussed elsewhere.

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A Notation and conventions

We use the "mostly positive" convention for the space-time signature, and typically omit symmetrized indices in tensor relations. Hence, our gamma matrices satisfy $\gamma^{0\,\dagger} = -\gamma^0, \quad \gamma^{i\,\dagger} = +\gamma^i$, $\gamma^0\gamma^{\mu\,\dagger}\gamma^0 = \gamma^\mu$. In addition, a "prime" always denotes a trace: $U\,'$ is thus the trace of U, while $U\,''$ is its double trace. A generic multiple trace, however, is denoted by a bracketed suffix, so that $U^{[n]}$ is the n-th trace of U. This notation is a very convenient method of streamlining the presentation, but it also results in an effective calculational procedure. This is especially true in flat space, but also in AdS backgrounds provided some care is clearly exercised with the resulting non-commuting covariant derivatives. To take full advantage of the compact notation, one needs to make repeated use of a number of identities, which reflect some simple combinatorics. These rest on our convention of working with symmetrized objects *not* of unit strength, which is convenient in this context but is not commonly used. For instance, given of vectors A_μ and B_ν, $A\,B$ here stands for $A_\mu\,B_\nu + A_\nu\,B_\mu$, without additional factors of two. The key identities are then:

$$\begin{aligned}
(\partial^p\,\varphi)\,' \quad &= \quad \Box\,\partial^{p-2}\,\varphi \;+\; 2\,\partial^{p-1}\,\partial\cdot\varphi \;+\; \partial^p\,\varphi\,' \,, \\
\partial^p\,\partial^q \quad &= \quad \binom{p+q}{p}\partial^{p+q} \,, \\
\partial\cdot(\partial^p\;\varphi) \quad &= \quad \Box\;\partial^{p-1}\;\varphi \;+\; \partial^p\;\partial\cdot\varphi \,, \\
\partial\cdot\eta^k \quad &= \quad \partial\,\eta^{k-1} \,, \\
\left(\eta^k\,\varphi\right)' \quad &= \quad [\,D \;+\; 2\,(s+k-1)\,]\;\eta^{k-1}\,\varphi \;+\; \eta^k\,\varphi\,' \,, \\
(U\;V)\,' \quad &= \quad U\,'\;V + U\;V\,' + 2\,U\cdot V \,, \\
\eta\;\eta^{n-1} \quad &= \quad n\,\eta^n \,.
\end{aligned} \tag{A.1}$$

As anticipated, the basic ingredient in these expressions is the combinatorics, which is simply determined by number of relevant types of terms on the two sides. Thus, for a pair of flat derivatives, $\partial\,\partial = 2\,\partial^2$ reflects the fact that, as a result of their commuting nature, the usual symmetrization is redundant precisely by the overall factor of two that would follow from the second relation. In a similar fashion, for instance, the last identity reflects the different numbers of terms generated by the naive total symmetrization of the two sides: $\binom{2n}{2}\times(2n-1)!!$ for the expression on the *l.h.s*, and $(2n+1)!!$ for the expression on the *r.h.s.*.

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